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MODULI SPACES OF EINSTEIN METRICS ON 4-MANIFOLDS

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In this note, we announce some results showing unexpected similarities between the moduli spaces of constant curvature metrics on 2-manifolds (the Riemann moduli space) and moduli spaces of Einstein metrics on 4-manifolds. Let $\mathscr{E}$ denote the moduli space of Einstein metrics of volume 1 on a compact, orientable 4-manifold M^4. If $\mathscr{M}_1$ denotes the space of smooth Riemannian metrics of volume 1 on M, endowed with a suitable smooth topology, then the diffeomorphism group $\mathscr{D}$ acts smoothly on $\mathscr{M}_1$ and $\mathscr{E}$ is the subspace of $\mathscr{M}_1/\mathscr{D}$ consisting of (equivalence classes) of Riemannian metrics g satisfying the Einstein condition $E(g) = \mathrm{Ric}(g) - (\lambda/4)g = 0$; here Ric denotes the Ricci curvature and λ the scalar curvature. It is well known that λ is a constant for Einstein metrics in dimension ≥ 3. Concerning the coarse structure of $\mathscr{E}$, it is known **[6]** that $\mathscr{E}$ consists of countably many components $\mathscr{C}$, each of which is locally the quotient of a finite-dimensional real-analytic Hausdorff variety by a compact group action. The scalar curvature variable $\lambda\colon \mathscr{C} \to \mathbf{R}$ is constant on $\mathscr{C}$.

There is a natural Riemannian metric on $\mathscr{M}_1$, the L^2 metric, defined as follows: for α, β symmetric 2-tensors in $T_g\mathscr{M}_1 \cong S^2(M)$, let $\langle\alpha, \beta\rangle = \int_M (\alpha(x), \beta(x))_g \, dv_g(x)$, where $(\ ,\ )_g$ is the inner product on $S^2(M)$ induced by g and dv_g is the volume form given by g. This induces a Riemannian metric on $\mathscr{M}_1/\mathscr{D}$ and thus a metric on the components $\mathscr{C}$ (since $\mathscr{C}$ is real-analytically path connected). Note however that the L^2 metric on $\mathscr{M}_1/\mathscr{D}$ is never locally complete (i.e. small metric balls are not complete).

In dimension 2, Einstein metrics are naturally considered to be metrics of constant scalar curvature. Thus, $\mathscr{E}$ is exactly the space of constant curvature metrics, or equivalently, the space of complex structures, on a closed oriented surface M^2. Of course, this has been widely studied, see e.g. **[5]**. In this case, the L^2 metric on $\mathscr{E}$ is known as the Weil-Petersson metric. One has the following basic trichotomy for the structure of $\mathscr{E}$:

(i) $\lambda > 0 \Rightarrow \mathscr{E} = \{\mathrm{pt}\}$, consisting of the unique constant curvature metric of volume 1 on $M = S^2$.

(ii) $\lambda = 0 \Rightarrow \mathscr{E} = SL(2, \mathbf{Z})\backslash SL(2, \mathbf{R})/SO(2)$ is the space of flat metrics on the torus. The Weil-Petersson metric is the complete, bi-invariant metric of finite volume on $\mathscr{E}$.

(iii) $\lambda < 0 \Rightarrow \mathscr{E}$ is the moduli space of hyperbolic metrics on a surface Σ_g of genus $g > 1$, and is the quotient of an open ball B^{6g-6} by the properly discontinuous action of the Teichmüller modular group Γ_g. $\mathscr{E}$

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