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THE DIAMETER AND THE MEAN DISTANCE OF A RIEMANNIAN MANIFOLD

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Let M be a compact Riemannian manifold. For x, $y \in M$, denote by $\rho(x,y)$ the distance between x and y, i.e., the infimum of lengths of continuous piecewise C^1 paths between x and y. (Notice that for geodesically complete manifolds there is always a path from x to y of length $\rho(x,y)$.) The invariants of M based on the distance ρ are called *metrical invariants*. Most of them are related to the geometry of M. Let us mention a few of them.

The *diameter*, $\operatorname{diam}(M)$, of M is the maximal value of $\rho(x,y)$ for x, $y \in M$. If two points x, y are at distance $\operatorname{diam}(M)$, they are called *antipodal*. Denote by

$$\bar{\rho}(x,M) = \frac{1}{\operatorname{Vol}(M)} \int_M \rho(x,y)\, dV(y) \tag{1}$$

the average distance of points in M from x. Then

$$\bar{\rho}(M) = \frac{1}{\operatorname{Vol}(M)} \int_M \bar{\rho}(x,M)\, dV(x) \tag{2}$$

is called the *mean distance* of M. These distances are important metric invariants of M, but, surprisingly enough, they have not been extensively investigated yet.

Denote by Δ the Laplace-Beltrami differential operator (Laplacian for short) on M. It is well known that the spectrum of Δ is a discrete set of real numbers, $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$. The first nontrivial eigenvalue $\lambda_1 = \lambda_1(M)$ of the Laplacian is related to many geometric properties of M. In this note we present inequalities between the diameter, or the mean distance of M, and $\lambda_1(M)$. For $\delta > 0$, let $B(x,\delta)$ denote the ball of radius δ centered at $x \in M$, and let

$$v(M,\delta) := \min_{x\in M} \operatorname{Vol}(B(x,\delta)).$$

THEOREM 1. *Let M be a compact Riemannian manifold with the first nontrivial eigenvalue of the Laplace operator on M equal to λ_1. Then for any $\delta > 0$,*

$$\operatorname{diam}(M) \le 2\left(\delta + \frac{2}{\sqrt{\lambda_1}} \left\lceil \log_2 \frac{\operatorname{Vol}(M)}{2v(M,\delta)} \right\rceil \right). \tag{3}$$

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