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HARMONIC MEASURE IN CONVEX DOMAINS

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Introduction. Let Ω be an open, convex subset of $\mathbf{R}^N$. At almost every point x of $\partial\Omega$, with respect to surface measure $d\sigma$, there is a unique outer unit normal θ. The map $g\colon \partial\Omega \to S^n$ given by $g(x) = \theta$ and defined almost everywhere is called the Gauss map. (S^n, $n = N-1$, is the unit sphere in $\mathbf{R}^N$.) Suppose that the origin 0 belongs to Ω. Harmonic measure for Ω at 0 is the probability measure ω such that for all continuous functions f on $\partial\Omega$[1],

$$u(0) = \int_{\partial\Omega} f\, d\omega$$

where u solves the Dirichlet problem: $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$.

Since Ω is a Lipschitz domain, Dahlberg's theorem **[4]** implies that $d\omega$ and $d\sigma$ are mutually absolutely continuous. Thus we can define a measure μ on S^n by $\mu = g_*\omega$ or

$$\mu(E) = \omega(g^{-1}(E)) \quad \text{for all } E \subseteq S^n.$$

We would like to pose the inverse problem: Given a probability measure μ on S^n, is there a domain Ω for which $\mu(E) = \omega(g^{-1}(E))$? Loosely speaking, we would like to find the convex domain given harmonic measure as a function of the unit normal.

We will solve the problem in case $d\mu = R\,d\theta$, R smooth and positive.

THEOREM 1. *For k an integer $\geq k(N)$ and $0 < \alpha < 1$, let $R \in C^{k,\alpha}(S^{N-1})$ be a positive function with $\int R\,d\theta = 1$. There exists a strictly convex domain Ω containing the origin, with $C^{k+2,\alpha}$ boundary, such that for $E \subset S^{N-1}$,*

$$\omega(g^{-1}(E)) = \int_E R\,d\theta,$$

where g is the Gauss map and ω is harmonic measure for Ω at 0. The domain Ω is unique up to dilation.

Our problem is natural for three reasons. First, it is analogous and closely related to the classical Minkowski problem. Second it is entirely solved in the plane by a continuous version of the well-known Schwarz-Christoffel formula. Third, the proof requires new, optimal estimates for

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[1]If Ω is unbounded, we suppose further that f tends to zero at infinity ($f \in C_0(\partial\Omega)$) and that u tends to zero at infinity.