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AMENABLE GROUP ACTIONS ON THE INTEGERS; AN INDEPENDENCE RESULT

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Issues involving the uniqueness of Lebesgue measure led to questions as to what extent a group G acting on a set X determines the structure (and number) of G-invariant finitely additive probability measures on X (G-*invariant means*). For example, Banach **[B]** showed that there was more than one rotation invariant finitely additive probability measure on the measurable subsets of S^1. More recently Sullivan[S] and independently Margulis **[M1, M2]** (for $n \geq 4$) and Drinfield **[D]** (for $n = 2, 3$) showed that Lebesgue measure is the unique finitely additive rotation invariant measure on the Lebesgue measurable subsets of S^n. An easier example is the following: Let μ be any two-valued finitely additive probability measure defined on all subsets of the natural numbers. Let G be the group of all permutations of **N** that are equal to the identity function on a set of μ-measure one. Then μ is the unique G-invariant finitely additive probability measure on $\mathscr{P}(\mathbf{N})$. Rosenblatt, noting that all of the known instances of uniqueness involved nonamenable groups, asked whether an amenable group G acting on a set X could uniquely determine a finitely additive invariant probability measure on X. (For amenable groups invariant measures always exist.) The main result of this note (Corollary to Theorem 3) is that in the concrete case of locally finite (hence amenable) groups acting on the natural numbers, the question of whether there is a G with a unique invariant mean is independent of the standard axioms for mathematics (Zermelo-Fraenkel set theory with the axiom of choice; ZFC).

We begin with some negative results. Let G act on a set X. Rosenblatt and Talagrand **[RT]** showed that if G is nilpotent, then G does not determine a unique invariant mean on X. Krasa **[K]** improved this to solvable groups.

Let N! be the group of permutations of the natural numbers with the topology of pointwise convergence. Recall that an *analytic* subset A of N! is a projection of a Borel set $B \subseteq \text{N!} \times \text{N!}$ onto the first coordinate. In particular any Borel set is analytic.

THEOREM 1. *If $G \subseteq$ N! is an analytic amenable group then G does not determine a unique invariant mean on $\mathscr{P}(\mathbf{N})$.*

COROLLARY. *No countable group determines a unique invariant mean.*

The idea of the proof is to show that if G has a unique invariant mean μ, then this mean is determined by G in a very concrete (positive $\Sigma^1_1(G)$)

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