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The homotopy index and partial differential equations, by Krzysztof P. Rybakowski. Universitext, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xii + 208 pp., \$39.50. ISBN 3-540-18067-2

"Toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis situs. J'avais besoin des données de cette science pour poursuivre mes études sur les courbes définies par les équations différentielles et pour les étendre aux équations différentielles d'ordre supérieur et en particulier à celles du problème des trois corps." This is the way Poincaré describes, in 1901, the role of topology in the genesis of the qualitative theory of differential equations, and the fruitful connection between the two areas has been of increasing importance since.

The basic concepts of general topology have found their use in the pioneering contribution of Birkhoff on dynamical systems considered as continuous flows in metric spaces, providing a unified setting for the various types of flows defined by differential equations, Volterra equations or iterates of mappings. We owe to Birkhoff the important notions of limit sets, invariant sets and recurrent motions. Recall that a *dynamical system* or *flow* on a metric space X is a continuous mapping $\pi: X \times \mathbf{R} \to X$ such that $\pi(\cdot, 0) = I_X$ and $\pi(\pi(x,t),s) = \pi(x,t+s)$ for all $x \in X$ and $t,s \in \mathbf{R}$. If, for each $x \in \mathbf{R}^n$, the Cauchy problem

$$y' = f(y), \qquad y(0) = x, \tag{1}$$

has a unique solution $y(t;x)$ defined for all $t \in \mathbf{R}$, then the mapping $\pi: (x,t) \mapsto y(t;x)$ is a dynamical system on $\mathbf{R}^n$. A subset $A \subset X$ is said to be an *invariant set* for π if $\pi(t,x) \in A$ when $t \in \mathbf{R}$ and $x \in A$. Important examples of invariant sets for the dynamical system associated to (1) are its equilibria (i.e. the zeros of f) and its periodic orbits.

The relation of algebraic topology with the study of nonlinear differential equations is not less important and, already in 1883, Poincaré made explicit use of Kronecker's characteristic, the forerunner of Brouwer's topological degree, to prove the existence of periodic solutions in the three-body problem. This degree can also be used to study the zeros of a continuous vector field f on $\mathbf{R}^n$, i.e., the equilibria of the associated differential equations. Namely, given a bounded open set B with no zeros of f on its boundary, the Brouwer degree of f in B is an algebraic count of the number of zeros of f in B, and in particular is equal to zero when f does not vanish in B; it only depends on the behavior of f on the boundary ∂B of B and is insensitive to small perturbations of f (and hence to continuous deformations of f which do not introduce zeros on ∂B) (see e.g. **[4]** for precise definitions and extensions). When f is the gradient of some real function F on $\mathbf{R}^n$, the *Morse index* (see e.g. **[3]**) provides a sharper tool to analyze the zeros of f, i.e. the critical points of F, when they satisfy a nondegeneracy condition.