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Ergodic theory of random transformations, by Yuri Kifer. Progress in Probability and Statistics, vol. 10, Birkhäuser, Boston, Basel, Stuttgart, 1986, 210 pp., $34.00. ISBN 0-8176-3319-7

Traditionally ergodic theory has been the qualitative study of iterates of an individual transformation, of a one-parameter flow of transformations (such as that obtained from the solutions of an autonomous ordinary differential equation), and more generally of a group of transformations of some state space. Usually ergodic theory denotes that part of the theory obtained by considering a measure on the state space which is invariant or quasi-invariant under the group of transformations. However in 1945 Ulam and von Neumann pointed out the need to consider a more general situation when one applies in turn different transformations chosen at random from some space of transformations. Considerations along these lines have applications in the theory of products of random matrices [**2**, **3**], random Schrödinger operators [**2**], stochastic flows on manifolds [**6**], and differentiable dynamical systems.

Mathematically the set up is as follows. Let M be a space, $\mathscr{B}$ a σ-algebra of subsets of M and let $\mathscr{T}$ be a collection of measurable transformations of M into M. For example, if M is a topological space we could choose $\mathscr{T}$ to be the space, $C(M, M)$, of all continuous transformations of M into M, and if M is a smooth manifold we could take $\mathscr{T}$ to the space, $D(M, M)$ of all smooth transformations of M into M. Suppose $\mathscr{T}$ is equipped with a σ-algebra so that the map $(f, x) \to f(x)$ of $\mathscr{T} \times M \to M$ is measurable. Let m be a probability measure on $\mathscr{T}$. We want to study the action on M of compositions of elements of $\mathscr{T}$ chosen independently with distribution m. So consider the direct product space $\Omega = \mathscr{T}^{\mathbb{N}}$ equipped with the direct product measure $p = m^{\mathbb{N}}$, where $\mathbb{N}$ denotes the natural members. The elements of Ω are sequences $w = (w_1, w_2, w_3, \dots)$ of members of $\mathscr{T}$. There is a natural transformation, $S: \Omega \to \Omega$, of Ω called the shift map and defined by $S((w_1, w_2, w_3, \dots)) = (w_2, w_3, \dots)$. The shift preserves the probability p (*i.e.* $p(S^{-1}A) = p(A)$ for every measurable subset A of Ω) and p is ergodic for S (*i.e.* if A is a measurable subset of Ω and $S^{-1}A = A$ then $p(A) = 0$ or 1). Consider the skew-product transformation $T: \Omega \times M \to \Omega \times M$ defined by $T(w, x) = (Sw, w_1(x))$ where $w = (w_1, w_2, \dots) \in \Omega$ and $x \in M$. Iterating gives $T^n(w, x) = (S^n w, w_n \circ w_{n-1} \circ \cdots \circ w_1(x))$ for $n \geq 1$, and the second coordinate gives the action of the randomly chosen maps on M.

This induces on M a discrete-time Markov Process with the probability, $P(x, B)$, of moving from the point $x \in M$ to a point of the measurable subset $B \subset M$ in one unit of time given by $P(x, B) = m(\{f \in \mathscr{T} | f(x) \in B\})$. For some applications, such as stochastic stability of diffeomorphisms [**5**], it seems more natural to consider certain Markov processes on M rather than actions by random maps, so one should consider which Markov