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ETA-INVARIANTS AND VON NEUMANN ALGEBRAS

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1. Main theorem. Let M be a compact oriented Riemannian manifold of dimension $4k-1$. The operator $D = *d + d^*$ acting on the $2k-1$ forms on M is selfadjoint, and for any Hermitian vector bundle $E \to M$ with connection ∇, there is a selfadjoint operator $D \otimes \nabla$ acting on the smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E$. The data (D, E) defines a class $[D, E]$ in the odd analytic K-homology group $K_1^{an}(M)$. We develop in this work an equality between two methods of pairing $[D, E]$ with real-valued K^1-cohomology classes.

Let $\Gamma = \pi_1(M)$ and $A : \Gamma \to U_N$ be a representation, which determines a flat principal U_N-bundle $P_A \to M$ and an associated flat C^N-vector bundle $E_A \to M$. If there exists a bundle trivialization $\theta : P_A \cong M \times U_A$, then it is well known that the pair (A, θ) represents a class $\overline{A}$ in $K^1(M) \otimes R$, (§5, **[APS 2]**). The first pairing of $[D, E]$ and $\overline{A}$ uses the relative eta-invariant to define the flat bundle index, as in (§5, ibid). Let ∇^0 be the flat connection on $M \times U_N$ associated to the product structure, and let ∇^1 be the flat connection associated with the push-forward under θ of the flat connection on P_A. Define a smooth, one-parameter family of selfadjoint operators on smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E \otimes C^N$ by $D_t = t \cdot D \otimes \nabla \otimes \nabla^1 + (1-t) \cdot D \otimes \nabla \otimes \nabla^0$. The eta-invariant $\eta(D_t)$ is smooth as a function of t except for a finite number of bounded jump discontinuities, so there exists a well-defined continuous derivative function $\eta(D_t)'$ **[APS 1,2]**. Define

$$\langle [D, E], \overline{A} \rangle_\eta = \int_0^1 \eta(D_t)' \, dt. \tag{1}$$

The second pairing uses the von Neumann algebra $W^*(\Gamma)$ associated to the universal covering $\tilde{M}$ of M **[A 1]**. The lift to $\tilde{M}$, $\widetilde{D \otimes \nabla}$, of the operator $D \otimes \nabla$ is essentially selfadjoint, and we introduce the projection, P^+, from $H = L^2(\tilde{M}, \Lambda^{2k-1}(T^*M) \otimes E \otimes C^N)$ onto the positive space H^+ of $D \otimes \nabla$. The operator P^+ is Γ-invariant, so defines an element $[P^+]$ in $K_1(W^*(\Gamma))$. The data (A, θ) defines a map $u : \tilde{M} \to U_N$ by restricting the composition $p_2 \circ \theta : P_A \to M \times U_N \to U_N$ to the leaf of F_A on P_A through a basepoint, where F_A is the U_N-invariant foliation of P_A associated to the flat structure. We pair $[P^+]$ to $[u] \in K^1_{(\infty)}(M)$ by constructing an associated Toeplitz operator, which is Fredholm in the sense of Breuer, and has a von Neumann index which is the continuous dimension of the

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