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CONTROLLED TOPOLOGY IN GEOMETRY

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The purpose of the present note is to announce some finiteness theorems for classes of Riemannian manifolds (cf. A, B and D below).

Let $\mathscr{M}_{k,d,v}^{K,D,V}(n)$ denote the class of closed Riemannian n-manifolds with sectional curvatures between k and K, diameter between d and D, and volume between v and V. Here $k \leq K$ are arbitrary, $0 \leq d \leq D$, and $0 \leq v \leq V$.

THEOREM A. *For $n \neq 3, 4$ the class $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ contains at most finitely many diffeomorphism types.*

This unifies and generalizes the following two theorems in high dimensions.

THEOREM (J. CHEEGER [C, P]). *The Class $\mathscr{M}_{k,0,v}^{K,D,\infty}(n)$ contains at most finitely many diffeomorphism types.*

THEOREM (K. GROVE, P. PETERSEN [GP]). *The class $\mathscr{M}_{k,0,v}^{\infty,D,\infty}$ contains at most finitely many homotopy types.*

For $k > 0$ and $n = 3$, the conclusion in Theorem A follows by Hamilton's theorem in [H]. For $k > 0$ and $n = 4$ the fundamental group is either trivial or $\mathbf{Z}_2$ by Synge's theorem. Using Freedman's classification of simply connected topological 4-manifolds together with the above theorem and standard surgery theory then yields (cf. also [HK]).

COROLLARY B. *For $k > 0$ the class $\mathscr{M}_{k,0,v}^{\infty,\infty,\infty}(n)$ contains at most finitely many diffeomorphism (resp. homeomorphism) types when $n \neq 4$ (resp. $n = 4$).*

The basic construction in [GP] exhibits for each $M \in \mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ a suitable strong deformation retraction of an *a* priori neighborhood of the diagonal in $M \times M$ onto the diagonal. This enables one to find $R, C > 0$ so that for all $p \in M$ the metric r-ball $B(p, r)$ is contractible inside $B(p, C \cdot r)$ whenever $r \leq R$. This latter property carries over to any compact space $X = \lim M_k$ in the Gromov-Hausdorff closure of $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$, moreover $\dim X \leq n$, cf. [PV]. Using the local contractibility properties, rather than the deformations as in [GP], one gets homotopy equivalences

$$M_k \underset{g_k}{\overset{f_k}{\rightleftarrows}} X$$

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