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COMPACT MANIFOLDS WITH A LITTLE NEGATIVE CURVATURE

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1. Bochner's Theorem states that a compact oriented Riemannian manifold (M, g) with positive Ricci curvature has $H^1(M;\mathbf{R}) = 0$. Myers' Theorem implies the stronger result that $\pi_1(M)$ is finite under the same hypothesis. Both theorems fail if the Ricci curvature is positive except on a set of arbitrarily small diameter, since every compact manifold admits such a metric of volume one. Nevertheless, we can extend these theorems and the Bochner Theorem for p-forms, yielding topological obstructions to manifolds admitting metrics with a little negative curvature.

2. Results for $H^1(M;\mathbf{R})$. The Laplacian on p-forms has the Weitzenböck decomposition $\Delta^p = \nabla^*\nabla + R^p$; here ∇ is the Levi-Civita connection and $R^p \in \mathrm{End}(\Lambda^p T^*M)$ with $R^1 = \mathrm{Ricci}$. We write $R^p(x) \geq R_0$ for $x \in M$ if the lowest eigenvalue of $R^p(x)$ is at least R_0. In what follows, we normalize all metrics to have volume one.

THEOREM 1. *Pick $R_0 > 0$ and $K < 0$. There exists $\varepsilon = \varepsilon(R_0, K, \dim M) > 0$ such that if $\mathrm{Ric}(x) \geq R_0$ except on a set A, with diameter $\mathrm{diam}(A) \leq \varepsilon$, where $\mathrm{Ric}(x) \geq K$, then $H^1(M;\mathbf{R}) = 0$.*

In other words, if the metric has a deep well of negative Ricci curvature, we may still conclude $H^1(M;\mathbf{R}) = 0$ provided the well is narrow enough. Notice that there is no restriction on the topology of A.

Theorem 1 is a consequence of the following weaker version about metrics with a shallow well of negative Ricci curvature.

THEOREM 1′. *Pick $R_0 > 0$. There exists $\varepsilon' = \varepsilon'(R_0, \dim M) > 0$ and $\delta = \delta(R_0, \dim M) < 0$ such that if $\mathrm{Ric}(x) \geq R_0$ except on a set A, with $\mathrm{diam}(A) \leq \varepsilon'$, where $\mathrm{Ric}(x) \geq \delta$, then $H^1(M;\mathbf{R}) = 0$.*

We sketch a proof of Theorem 1′. By semigroup domination for the heat flow on one forms, it is enough to show that $\Delta^0 + \mathrm{Ric}' > 0$, where $\mathrm{Ric}'(x)$ is the lowest eigenvalue of Ricci at x. By an elementary argument, we have

LEMMA 2. *Let $V\colon M \to \mathbf{R}$ be continuous. If (i) $\int_M V\, d\mathrm{vol}(g) > 0$ and (ii) $\lambda_1 \geq -V_{\min} + \frac{\|V - V_{av}\|^2}{\int_M V}$, then $\Delta^0 + V > 0$.*

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