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ON REAL ALGEBRAIC MODELS OF SMOOTH MANIFOLDS

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An affine nonsingular real algebraic variety X diffeomorphic to a smooth manifold M is said to be an *algebraic model* of M. The remarkable theorem of Nash-Tognoli asserts that each compact smooth manifold M has an algebraic model [**16** or **6**, Theorem 14.1.10]. In fact, there exists an infinite family $\{X_i\}_{i\in\mathbf{N}}$ of irreducible algebraic models of M such that X_i and X_j are birationally nonisomorphic for $i \neq j$ [**10**] (cf. also [**7**] for a proof in a special case). In view of these results, it seems natural and interesting to investigate algebro-geometric properties of various algebraic models of a given smooth manifold. This paper addresses a few questions of this type. For notions and results of real algebraic geometry we refer the reader to the book [**6**].

Given a compact affine nonsingular real algebraic variety X, denote by $H_k^{\mathrm{alg}}(X, \mathbf{Z}/2)$ the subgroup of $H_k(X, \mathbf{Z}/2)$ of the homology classes represented by (Zariski closed) k-dimensional algebraic subvarieties of X [**6**, Chapter 11 or **11**]. Let $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$ be the image of $H^{\mathrm{alg}}_{d-k}(X, \mathbf{Z}/2)$, $d = \dim X$, under the Poincaré duality isomorphism $H_{d-k}(X, \mathbf{Z}/2) \to H^k(X, \mathbf{Z}/2)$. Although the groups $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$ are one of the most important invariants of X (a sample of applications can be found in [**1, 2, 3, 6, 8, 9**]), our knowledge of their behavior is still rather limited. Here we consider the following.

PROBLEM. Let M be a compact smooth manifold and let G be a subgroup of $H^k(M, \mathbf{Z}/2)$. When are there an algebraic model X of M and a diffeomorphism $\varphi: X \to M$ such that the induced isomorphism

$$\varphi^*: H^k(M, \mathbf{Z}/2) \to H^k(X, \mathbf{Z}/2)$$

maps G onto $H^k_{\mathrm{alg}}(X, \mathbf{Z}/2)$?

This problem has attracted the attention of several mathematicians (cf. [**3, 4, 5, 6, 12, 14, 15**]), however, the results are far from complete. We have a solution for $k = 1$, M connected, and $\dim M \geq 3$.

THEOREM 1. *Let M be a compact connected smooth manifold with* $\dim M \geq 3$ *and let G be a subgroup of $H^1(M, \mathbf{Z}/2)$. Then the following conditions are equivalent*:

(i) *There exists an algebraic model X of M and a diffeomorphism $\varphi: X \to M$ such that $\varphi^*(G) = H^1_{\mathrm{alg}}(X, \mathbf{Z}/2)$.*

(ii) *The first Stiefel-Whitney class $w_1(M)$ of M is in G.*

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