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VARIETIES IN FINITE TRANSFORMATION GROUPS

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ABSTRACT. The equivariant cohomology ring of a G-space X defines a homogeneous affine variety. Quillen [**Q**] and W. Y. Hsiang [**Hs**] have determined the relation between such varieties and the family of isotropy subgroups as well as their fixed point sets when $\dim X < \infty$. In modular representation theory, J. Carlson [**Cj**] introduced cohomological support varieties and rank varieties (the latter depending on the group algebra) and explored their relationship. We define rank and support varieties for G-spaces and G-chain complexes and apply them to cohomological problems in transformation groups. As a corollary, a useful criterion for $\mathbf{Z}G$-projectivity of the reduced total homology of certain G-spaces is obtained, which improves the projectivity criteria of Rim [**R**], Chouinard [**Ch**], and Dade [**D**].

1. Introduction. Let G be a finite group. Assume in the sequel that all modules, including total homologies of G-spaces and G-chain complexes, are finitely generated. In a fundamental paper [**R**], D. S. Rim proved that a $\mathbf{Z}G$-module M is $\mathbf{Z}G$-projective if and only if $M|\mathbf{Z}G_P$ is $\mathbf{Z}G_P$-projective for all Sylow subgroups $G_P \subseteq G$. This theorem has had many applications to local-global questions in topology, algebra, and number theory. In his thesis [**CH**] Chouinard greatly improved Rim's theorem by proving that the $\mathbf{Z}G$-projectivity of M is detected by restriction to p-elementary abelian subgroups $E \subseteq G$, i.e. $E \cong (\mathbf{Z}/p\mathbf{Z})^n = \langle x_1, \dots, x_n \rangle$. If M is $\mathbf{Z}$-free (a necessary condition for projectivity), it suffices to consider $k \otimes M$, where $k = \bar{\mathbf{F}}_p$ when restricting to E. In a deep and difficult paper [**D**], Dade provided the ultimate criterion: A kE-module M is kE-free if and only if for all $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$, the units $u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1)$ of kE act freely on M. Thus the projectivity question reduces to the restrictions to all p-order cyclic subgroups $\langle u_\alpha \rangle \subseteq kG$. Since $k = \bar{\mathbf{F}}_p$, all but finitely many are not subgroups of G. When the $\mathbf{Z}G$-module M arises as the homology of a G-space, we have a much simpler criterion which is a natural sequel to Dade's theorem.

THEOREM 1. *Let X be a connected paracompact G-space (possibly $\dim X = \infty$), and let $M = \bigoplus \bar{H}_i(X)$ with induced G-action. Assume that for each maximal $A \cong (\mathbf{Z}/p\mathbf{Z})^n \subseteq G$, the Serre spectral sequence of the Borel construction*

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