BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

# CLASSIFICATION OF INVARIANT CONES IN LIE ALGEBRAS

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All *Lie algebras* in the following are finite dimensional real Lie algebras. A *cone* in a finite dimensional real vector space is a closed convex subset stable under the scalar multiplication by the set $\mathbf{R}^+$ of nonnegative real numbers; it is, therefore additively closed and may contain vector subspaces. A cone $W$ in a Lie algebra $\mathfrak{g}$ is called *invariant* if

$$e^{\operatorname{ad} x}(W) = W \quad \text{for all } x \in \mathfrak{g}. \tag{1}$$

We shall describe invariant cones in Lie algebras completely. For simple Lie algebras see [**KR82**, **Ol81**, **Pa84**, and **Vi80**].

Some observations are simple: *If $W$ is an invariant cone in a Lie algebra $\mathfrak{g}$, then the edge $\mathfrak{e} = W \cap -W$ and the span $W - W$ are ideals.* Therefore, if one aims for a theory without restriction on the algebra $\mathfrak{g}$ it is no serious loss of generality to assume that $W$ is *generating*, that is, satisfies $\mathfrak{g} = W - W$. This is tantamount to saying that $W$ has inner points. Also, the homomorphic image $W/\mathfrak{e}$ is an invariant cone with zero edge in the algebra $\mathfrak{g}/\mathfrak{e}$. Therefore, nothing is lost if we assume that $W$ is *pointed*, that is, has zero edge. Invariant pointed generating cones can for instance be found in $\mathrm{sl}(2, \mathbf{R})$, the oscillator algebra and compact Lie algebras with nontrivial center (see [**HH85b**, **c**, **HH86a**, or **HHL87**]).

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be *compactly embedded* if the analytic group $\operatorname{Inn}_{\mathfrak{g}} \mathfrak{h}$ generated by the set $e^{\operatorname{ad} \mathfrak{h}}$ in $\operatorname{Aut} \mathfrak{g}$ has a compact closure. Even for a compactly embedded Cartan algebra $\mathfrak{h}$ of a solvable algebra $\mathfrak{g}$, the analytic group $\operatorname{Inn}_{\mathfrak{g}} \mathfrak{h}$ need not be closed in $\operatorname{Aut}_{\mathfrak{g}}$ [**HH86**]. An element $x \in \mathfrak{g}$ is called *compact* if $\mathbf{R} \cdot x$ is a compactly embedded subalgebra, and the set of all compact elements of $\mathfrak{g}$ will be denoted $\operatorname{comp} \mathfrak{g}$. It is true, although not entirely superficial that a *superalgebra is compactly embedded if and only if it is contained in* $\operatorname{comp} \mathfrak{g}$.

1. THEOREM (THE UNIQUENESS THEOREM [**HH86b**]). *Let $W$ be an invariant pointed generating cone in a Lie algebra $\mathfrak{g}$. Then*

(i) $\operatorname{int} W \subseteq \operatorname{comp} \mathfrak{g}$.

(ii) *If $H$ is any compactly embedded Cartan algebra, then*

(a) $H \cap \operatorname{int} W \neq \emptyset$, *and*

(b) $\operatorname{int} W = (\operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}) \operatorname{int}_{\mathfrak{h}}(\mathfrak{h} \cap W)$.

*In particular, compactly embedded Cartan algebras exist, and if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are compactly embedded Cartan algebras and $W_1$ and $W_2$ are invariant pointed generating cones of $\mathfrak{g}$ such that $\mathfrak{h} \cap W_1 = \mathfrak{h} \cap W_2$, then $W_1 = W_2$.* □

Received by the editors December 16, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E60, 22E15.


0273-0979/88 $1.00 + $.25 per page