BOOK REVIEWS

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Regular variation, by N. H. Bingham, C. M. Goldie, and J. L. Teugels. Encyclopedia of Mathematics and Its Applications, vol. 27, Cambridge University Press, Cambridge, 1987, xix + 491 pp., $75.00. IBN 0-521-30787-2

Regular variation can be seen initially as an attempt to define the derivative of a real function ϕ at infinity. Write the differential quotient $\{\phi(t+h) - \phi(t)\}/h$ for $h \neq 0$. Now instead of keeping t fixed and letting $h \to 0$, we keep h fixed and let $t \to \infty$. If ϕ is (Borel-) measurable and the limit exists for all $h \neq 0$, then this limit does not depend on h (since the limit of $\phi(t+h) - \phi(t)$ satisfies Cauchy's functional equation). Moreover there exists a differentiable function ϕ_0 such that $\phi_0(t) - \phi(t) \to 0$ $(t \to \infty)$ and

$$\lim_{t\to\infty} \phi_0'(t) = \lim_{t\to\infty} \frac{\phi(t+h)-\phi(t)}{h}.$$

If $f := \exp \circ \phi \circ \log$, then $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ is measurable and the property above translates into

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha \quad \text{for all } x > 0; \tag{1}$$

here α is a real parameter. This is the definition of regular variation.

It turns out that many properties that hold identically for power functions, hold asymptotically for functions of regular variation. For example the relation

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{tf(t)} \int_0^t f(s)\,ds &= \lim_{t\to\infty} \int_0^1 \frac{f(tx)}{f(t)}\,dx \\ &= \int_0^1 \lim_{t\to\infty} \frac{f(tx)}{f(t)}\,dx = \int_0^1 x^\alpha\,dx = \frac{1}{1+\alpha}\end{aligned} \tag{2}$$

holds whenever the integrals are finite. In fact relation (2) characterizes regular variation (except for the integrability), i.e., a regularly varying function is precisely a function that is asymptotically of the same order as its average. Relation (2) suggests that there should be some automatic uniformity in relation (1) and indeed this is true on compact x-subsets of $(0,\infty)$. The second equality in (2) also holds with the integration interval $[0,1]$ replaced by $[1,\infty)$. A generalization of both is

$$\lim_{t\to\infty} \frac{1}{tf(t)} \int_0^\infty k(x/t) f(x)\,dx = \lim_{t\to\infty} \int_0^\infty k(x) \frac{f(tx)}{f(t)}\,dx = \int_0^\infty k(x) x^\alpha\,dx. \tag{3}$$