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# TIGHTLY CLOSED IDEALS

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All rings are Noetherian, commutative, with 1, and contain a field. We define the tight closure of an ideal in characteristic $p$ and for algebras essentially of finite type over a field, and use it to give new, much simpler proofs of many theorems in a greatly strengthened form, including the result that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay (C-M), the Briançon-Skoda theorem, and the monomial conjecture. Although we do not define tight closure in arbitrary rings, many of our results can be proved in that generality by using Artin approximation to reduce to the affine case. Results discussed here are treated in full in [**HH**].

**1. Tight closure.** Let $R^0 = R - \bigcup\{P : P \text{ is a minimal prime of } R\}$. Let $I \subseteq R$ be an ideal. If $\operatorname{char} R = p > 0$ we say that $x \in R$ is in the tight closure, $I^*$, of $I$, if there exists $c \in R^0$ such that for all $e \gg 0$, $cx^{p^e} \in I^{[p^e]}$, where $I^{[q]} = (i^q : i \in I)$ when $q = p^e$. If $R$ is of finite type over a field $K$ of characteristic 0, we say that $x \in I^*$ if there exist $c \in R^0$, a finitely generated $\mathbb{Z}$-subalgebra $D \subseteq K$, a finitely generated $D$-flat $D$-subalgebra $R_D$ of $R$, and an ideal $I_D \subseteq R_D$ such that $R \cong K \otimes_D R_D$, $I = I_D R$, and for all maximal ideals $m$ of $D$, if $\kappa = R/m$ (with the subscript $\kappa$ denoting images after applying $\kappa \otimes_D$), and $p = \operatorname{char} \kappa$, then $c_\kappa x_\kappa^q \in I_\kappa^{[q]}$ (in $R_\kappa \cong R_D/mR_D$) for all $q$ of the form $p^e$, $e \gg 0$. If $R$ is essentially of finite type over a field $K$ of char 0 we define $I^*$ as $\bigcup_B (I \cap B)^*$ as $B$ runs through all subrings of $R$ of finite type over $K$ such that $R$ is a localization of $B$.

These intricate definitions yield an immensely powerful tool. We note that $I \subseteq I^* = I^{**}$ and that $I^* \subseteq \overline{I}$, the integral closure of $I$, but that $I^*$ is usually much smaller than $\overline{I}$. If $I = I^*$, we call $I$ tightly closed. A key point is that if $R$ is regular, then $I = I^*$ for all $I$. Suppose that $R$ is regular of $\operatorname{char} p > 0$. To see that $I = I^*$ we may assume that $(R, m)$ is local and that $y \in I^* - I$ where $I = (x_1, \ldots, x_m)R$. Then for some $c \in R^0$, $cy^q \in I^{[q]}$ for all $q = p^e \gg 0$. Since the Frobenius $f$ is flat, $I^{[q]} : y^q = (I : y)^{[q]}$, which implies that $c \in m^q$, and since this is true for every $q \gg 0$, $c = 0$, a contradiction. We call rings such that in all localizations $I = I^*$ for all $I$ $F$-regular.

(1.1) REMARK. If $R \subseteq S$ are domains where $*$ is defined, $S$ is regular, and $I$ is an ideal of $R$ then $I^*S = IS$, since $IS$ is tightly closed in $S$. Hence, $I^*$ is contained in all ideals containing $I$ which are contracted from regular overrings. We do not know whether $I^*$ is the intersection of all ideals containing $I$ which are contracted from regular overrings.

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