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NUMERICAL METHODS FOR EXTREMAL PROBLEMS IN THE CALCULUS OF VARIATIONS AND OPTIMAL CONTROL THEORY

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Introduction. New, general methods are given to find numerical solutions for extremal problems in the calculus of variations and optimal control theory. Theoretical methods are derived and used to establish pointwise a priori error estimates with maximum error at the node point, $\|e\|_\infty$, equal to $O(h^2)$ and a Richardson error of $O(h^4)$. This is done under the weak assumption that there are no conjugate points on the interval and not under the usual convexity assumptions.

Of practical interest is that these methods (i) are very easy to implement, (ii) hold for well-defined mixtures of initial value and boundary value problems, (iii) use multipliers, and not ill-conditioned penalty methods, for both equality and inequality constraints, in a natural, efficient manner, and (iv) are applicable to transversality, type-minimal time problems.

The heart of these methods is the algorithm (4) and the a priori estimates in Theorem 2 for the m-dependent variable problem in the calculus of variations given below. Once this is established we quote Hestenes [**5**] and show that very general optimal control problems can be easily reformulated and solved as calculus of variations problems.

The calculus of variations problem. The problem is to find numerical solutions for extremal solutions of

$$I(x) = \int_a^b f(t, x, x')\, dt, \tag{1}$$

where $x(t)$ is an m-vector. This will be done by finding approximate numerical solutions of the first variational problem

$$I'(x, y) = \int_a^b (y^T f_x + y'^T f_{x'})\, dt = 0 \tag{2}$$

for numerical admissible variations $y(t)$. The setting and background is given in Hestenes [**5**, pp. 57-62]. In particular, we require that the $m \times m$ matrix $f_{x'x'}$ be invertible for each t in $[a, b]$, enough smoothness on f to yield a unique piecewise smooth solution, and that (1) have no conjugate points in $[a, b]$.

Letting $\pi = (a = a_0 < a_1 < \cdots < a_N = b)$ be a partition of $[a, b]$, with $a_{k+1} - a_k = h = (b-a)/N$, and $z_k(t)$ the spline hat functions with $z_k(a_k) = 1$,

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