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ON THE DECIDABILITY OF DIOPHANTINE PROBLEMS IN COMBINATORIAL GEOMETRY

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ABSTRACT. In spite of Matiyasevic's solution to Hilbert's 10th problem some fifteen years ago it is still unknown whether there exists an algorithm to decide the solvability of diophantine equations within the field of rational numbers. In this note we show the equivalence of this problem with a conjecture of B. Grünbaum [**6**] on rational coordinatizability in combinatorial geometry. Such an algorithm exists if and only if the rational realizability problems for matroids, oriented matroids, and convex polytopes (Steinitz problem) are decidable.

1. Introduction and statement of the result. Many realizability problems in combinatorial and computational geometry can be formulated in terms of polynomial equations and inequalities with integer coefficients, and so these problems are decidable over the real numbers by a well-known result of Tarski [**14**].

The situation is different if we focus our attention on solutions in the field Q of *rational* numbers. In view of Matiyasevic's negative solution [**9**] to Hilbert's 10th problem in 1971, B. Grünbaum has conjectured [**6**, Conjecture 2.14] that there is no algorithm to enumerate all (isomorphism types of) arrangements of lines in the rational projective plane. In [**5**, p. 92] the same question has been raised for convex polytopes in rational Euclidean d-space, $d \geq 4$. Matiyasevic's result "there exists no algorithm to decide whether a system of diophantine equations has a solution among the rational *integers*" cannot be applied to prove Grünbaum's conjecture, and, as B. Mazur points out in a recent survey article [**10**], the corresponding problem for rational *numbers* is still open; see also Klee and Wagon [**8**].

In this note we show that Grünbaum's conjectures for line arrangements and convex polytopes as well as the corresponding conjecture for matroids are equivalent to the above problem. See White [**15**], Bachem [**1**], Bokowski and Sturmfels [**3**] and the references given there for the basic concepts of matroid theory and oriented matroids and [**13**] for recent results on irrational oriented matroids and polytopes.

THEOREM. *The following statements are equivalent.*

(1) *There exists an algorithm to decide for an arbitrary polynomial* $f \in Z[x_1, \ldots, x_n]$, $n \in N$, *whether* f *has zeros in the field* Q *of rational numbers.*

(2) *There exists an algorithm to decide for an arbitrary matroid* M *whether* M *is coordinatizable over* Q.

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