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STABLE HARMONIC 2-SPHERES IN SYMMETRIC SPACES

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A map $\phi: (M, g) \to (N, h)$ of Riemannian manifolds is *harmonic* if it extremizes the *energy* $E: C^\infty(M, N) \to R$ given (for compact M) by

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \mathrm{vol}.$$

A harmonic map ϕ is said to be *stable* if the second variation of E at ϕ is positive semidefinite. That is: for all smooth variations $\phi_t \in C^\infty(M, N)$ with $\phi_0 = \phi$ we have

$$d^2/dt^2 E(\phi_t)|_{t=0} \geq 0.$$

Of particular interest is the case where M is the sphere S^2 and N is a Riemannian symmetric space G/K. In this setting harmonic maps are branched minimal immersions, or the finite action solutions of the Euclidean nonlinear σ-model studied by physicists (see e.g. [**20**] and references cited therein). In the case G/K is Hermitian symmetric it follows from an argument of Lichnerowicz [**9**] that any holomorphic map is energy minimizing in its homotopy class and hence stable. The same is true of antiholomorphic (or $-$holomorphic) maps. The $\pm$holomorphic maps are the *instantons* of the nonlinear σ-model, and it is important to know if these are the only stable solutions. This is clearly not the case, as one sees by taking G/K to be a product of Hermitian symmetric spaces and by taking a map which is holomorphic into one factor and $-$holomorphic into the other. However, this is the only way a stable map can fail to be $\pm$ holomorphic, as the following theorem shows.

THEOREM 1. *Let $\phi: S^2 \to G/K$ be a stable harmonic map into an irreducible Hermitian symmetric space. Then ϕ is $\pm$holomorphic.*

This generalizes a result of Siu and Yau [**16**], who obtained Theorem 1 for the complex projective spaces as targets.

If the target G/K is a general symmetric space ϕ can always be lifted to a map into the simply connected covering space. A simply connected symmetric space then splits as a product of irreducible spaces with ϕ given by a harmonic map into each factor. As noncompact factors have nonpositive curvature the component of ϕ going into such a factor must be constant (by the results of Eells and Sampson [**2**], or more simply by the maximum principle [**4**]), which reduces us to the consideration of compact irreducible symmetric spaces. Moreover ϕ is stable if and only if all its components are. We can show that stable harmonic maps into irreducible compact Riemannian symmetric

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