BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

FUNCTIONAL ANALYSIS AND ADDITIVE ARITHMETIC FUNCTIONS

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In memory of Professor M. Kac

Geometry Prince has familiar,
Twin;
Upon looking out,
Sees himself looking in.

1. A function is *arithmetic* if it is defined on the positive integers. Those arithmetic functions which assume real values and satisfy $f(ab) = f(a) + f(b)$ for mutually prime integers a, b are classically called *additive*. The following examples illustrate the interest of these functions, both for themselves and for their applications.

An additive function is defined by its values on the prime powers. I shall denote a typical prime power by q, and the prime of which it is a power by q_0. A well-known additive function is $\omega(n)$ which, with $\omega(q) = 1$, counts the number of distinct prime divisors of n. Let $\nu_x(n; \ldots)$ denote the frequency amongst the positive integers not exceeding x of those for which property ... holds. Then as $x \to \infty$

$$\nu_x\left(n;\ \omega(n) - \log\log x \leqslant z(\log\log x)^{1/2}\right) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

More generally, for an additive function $f(n)$ define

$$A(x) = \sum_{q \leqslant x} \frac{f(q)}{q}\left(1 - \frac{1}{q_0}\right), \qquad B(x) = \left(\sum_{q \leqslant x} \frac{|f(q)|^2}{q}\left(1 - \frac{1}{q_0}\right)\right)^{1/2} \geqslant 0.$$

Then if $|f(q)| \leqslant 1$ for all q and $B(x)$ is unbounded with x, the celebrated theorem of Erdős-Kac [**18, 19**] asserts (essentially) that, as $x \to \infty$,

$$\nu_x(n;\ f(n) - A(x) \leqslant zB(x)) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

Received by the editors March 12, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11K65, 11N37, 47B99.
Partially supported by N.S.F. contract no. DMS-8500949.