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A GEOMETRIC APPROACH TO THE HYPERBOLIC JØRGENSEN INEQUALITY

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Introduction. Jørgensen's inequality [J] gives a necessary condition for two elements of $\mathrm{PSL}(2,\mathbf{C})$ to generate a nonelementary discrete group. If A and B are in $\mathrm{PSL}(2,\mathbf{C})$, the inequality says that

$$|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A,B] - 2| \geq 1, \tag{I}$$

where tr is the trace and [,] represents the commutator. This is one of the most useful and powerful tools available for determining nondiscreteness. The precise geometric meaning of this inequality has been unclear.

Here we first give an equivalent but more geometric formulation of the inequality (see (II)) in the case of hyperbolic elements in $\mathrm{PSL}(2,\mathbf{R})$. Next we outline a proof of the inequality for the case of purely hyperbolic subgroups of $\mathrm{PSL}(2,\mathbf{R})$ which shows that under these circumstances an even stronger inequality is satisfied (see (III)). For purely hyperbolic discrete groups (I) is trivially satisfied for all but the finite number of conjugacy classes of elements (or their inverses) with multiplier between 1 and $(3+\sqrt{5})/2$, whereas the stronger inequality is not.

Groups of Möbius transformations in space are an increasingly important area of study. A Jørgensen type inequality would be significant there, and it is hoped that this formulation of Jørgensen's inequality might point the way toward the proper formulation in space.

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The inequalities. Let A and B be hyperbolic matrices with multipliers R and K respectively. If v_A, w_A and v_B, w_B are the repelling and attracting fixed points of A and B respectively, let C be the cross ratio

$$((v_A - v_B)(w_A - w_B))/((v_A - w_B)(w_A - v_B)).$$

Note that C, R, and K are conjugation invariant.

Let $f(x)$ be the function $f(x) = x/(x-1)^2$ and note that $f(x) = f(1/x)$. Compute that $\mathrm{tr}[A,B] = 2 + f(C)/(f(R)f(K))$ (see [G]). Rewrite (I) as

$$\frac{1}{f(R)} + \frac{|f(C)|}{f(R)f(K)} \geq 1. \tag{II}$$

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