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COUNTING LATIN RECTANGLES

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A $k \times n$ Latin rectangle is a $k \times n$ array of numbers such that each row is a permutation of $\{1, 2, \ldots, n\}$ and each column has distinct entries. The problem of counting Latin rectangles is of considerable interest. Explicit formulas for $k = 3$ are fairly well known [**1–3**, **4**, pp. 284–286 and 506–507, **5**, **6**, **9–11**, **12**, pp. 204–210]. Formulas for $k = 4$ were found by Pranesachar et al. [**1**, **9**] and a complicated formula for all k was found by Nechvatal [**8**]. We give here a simple derivation of a formula similar to Nechvatal's. The formula implies that for fixed k, the number of $k \times n$ Latin rectangles satisfies a linear recurrence with polynomial coefficients. We use properties of the Möbius functions of partition lattices, as did Bogart and Longyear [**2**], Pranesachar et al. [**1**, **9**], and Nechvatal [**8**], but in a somewhat different way.

In order to state the formula, we first make some definitions. Let $\mathcal{P}$ be the set of partitions of $\mathbf{k} = \{1, 2, \ldots, k\}$ and let S be the set of nonempty subsets of $\mathbf{k}$. If f is a function from $\mathcal{P}$ to the nonnegative integers $\mathbf{N}$, and A is in S, then we set $\langle f, A\rangle = \sum_{\pi \ni A} f(\pi)$, where the sum is over all partitions π of which A is a block. We shall say that two functions $f, g \colon \mathcal{P} \to \mathbf{N}$ are *compatible* if $\langle f, A\rangle = \langle g, A\rangle$ for each A in S.

THEOREM. *The number of $k \times n$ Latin rectangles is*

$$\sum_{f,g} \frac{n!^2}{\prod_{\pi \in \mathcal{P}} f(\pi)! g(\pi)!} \prod_{A \in S} (-1)^{\langle f,A\rangle(|A|-1)} (|A| - 1)!^{\langle f,A\rangle} \langle f, A\rangle!,$$

where the sum is over all compatible pairs f, g of functions from $\mathcal{P}$ to $\mathbf{N}$ satisfying $\sum_{\pi \in \mathcal{P}} f(\pi) = \sum_{\pi \in \mathcal{P}} g(\pi) = n$.

PROOF. We first restate the problem in terms of bipartite graphs. Given a $k \times n$ "rectangle" satisfying the row conditions, but with column entries not necessarily distinct, we may associate to it a bipartite graph with vertex sets $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, and with edges colored in k colors. (We identify the set of colors with $\mathbf{k}$.) If the rectangle has the entry l

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