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THE ASYMPTOTICS OF $e^{P(z)}$ AND THE NUMBER OF ELEMENTS OF EACH ORDER IN S_n

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ABSTRACT. We answer a question that was raised in 1952 by Chowla, Herstein and Scott, concerning the asymptotic behavior of the number of elements of order m in the symmetric group S_n, for fixed m, as $n \to \infty$. The methods used include Hayman's method for asymptotics of coefficients of analytic functions, and the Lagrange inversion formula. The question had previously been answered only for prime m.

1. Introduction. In 1952 Chowla, Herstein and Scott [**2**] asked for the asymptotic behavior, for large n, of the number $f(m,n)$ of solutions of the equation $x^m = 1$ in the symmetric group S_n. They found the generating function and some recurrence relations for the $f(m,n)$.

In 1955 Moser and Wyman [**4**] found the answer when $m = 2$, i.e., they counted the involutions in S_n, for large n. Then they developed a method [**5**] that permitted them to solve the problem when $m = p$, a prime number.

The problem is discussed further in Bender [**10**], §8.1.

In this paper we will give an explicit answer that is valid for every m.

The ingredients of the solution are the following.

1° *Hayman's method.* In 1956 W. K. Hayman developed a general method for finding the asymptotic behavior of the coefficients of analytic functions (some later developments of the theory are in [**11, 12**]). This method has already had a number of applications to combinatorial problems. Hayman's machinery allows us to take the first step in the solution of the present problem.

The unfinished business that it leaves is that the answer is expressed in terms of a root of a certain equation, and that root must be determined with considerable accuracy in order to get an explicit asymptotic result.

2° *Lagrange's inversion formula.* We will use the famous inversion formula of Lagrange to find the solution of the polynomial equation referred to above. Remarkably, what it gives is the root expressed as an infinite series that is at once convergent and asymptotic.

The unfinished business that it leaves is that the coefficients of the series in question are given implicitly, but we want an explicit solution.

3° *Special properties.* Until this point the analysis will have been quite general, and applicable to any problem of the type considered. To get explicit

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