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THE TRACE FORMULA FOR VECTOR BUNDLES

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Let X be a compact Riemannian manifold and let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the spectrum of the Laplace operator. By a theorem of Hermann Weyl the spectral counting function

$$N(\lambda) = \#\{\lambda_i^2 < \lambda\} \tag{I}$$

satisfies a growth estimate of the form $O(\lambda^N)$, so its Fourier-Stieltjes transform

$$\int e^{i\lambda t}\, dN(\lambda) = \sum_k e^{\pm i\sqrt{\lambda_k t}} \tag{II}$$

is a tempered distributional function of t. The classical trace formula says that the singular support of (II) is contained in the length spectrum of X. Moreover, under suitable hypotheses on geodesic flow, the trace formula gives considerable information about the singularities in (II). (See [**DG** and **C**].)

There is a fairly straightforward (and not terribly interesting) generalization of the trace formula to vector bundles. (See, for instance, the introduction to [**DG**].) We will be concerned in this article with a much more subtle generalization inspired by recent articles of Hogreve, Potthoff, and Schrader [**HPS**], and Schrader and Taylor [**ST**] in *Communications in Mathematical Physics.*

Let G be a compact Lie group and $\pi\colon P \to X$ a principle G-bundle with connection. Given a finite-dimensional unitary representation, ρ, of G we will denote by $E\rho$ the vector bundle over X associated with ρ and by D_ρ the associated connection.

Now consider a *ladder* $\{\rho_e,\ e = 1, 2, \ldots\}$ of irreducible representations of G. (This means that the maximal weight of ρ_e is e times the maximal weight of ρ_1.) For given e let

$$\lambda_{k,e}, \qquad k = 1, 2, 3, \ldots$$

be the spectrum of the Laplace operator on $C^\infty(E\rho_e)$:

$$\Delta_e = D^*\rho_e D\rho_e + e^2.$$

The Hogreve-Potthoff-Schrader and Schrader-Taylor papers are concerned with asymptotic properties of the quantities e and $E = \lambda_{k,e}$ when e and E tend to infinity in such a way that the ratio $r = e\colon \sqrt{E}$ is (approximately) constant. One way to measure such asymptotic behavior is as follows: Fix a Schwartz function of one variable, $\varphi(s)$, with $\varphi(s) \geq 0$ and $\int \varphi(s)\, ds = 1$, and form the sum

$$N_{\varphi,r}(\lambda) = \sum_e \sum_{\lambda_{k,e} < \lambda^2} \varphi(r\sqrt{\lambda_{k,e}} - e). \tag{III}$$

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