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FREUD'S CONJECTURE FOR EXPONENTIAL WEIGHTS

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1. Freud's Conjecture. Let $W(x)$ be nonnegative on $\mathbf{R}$, positive on a set of positive Lebesgue measure, and let $W^2(x)$ have all moments finite. Then we call W a *weight function.* Associated with $W^2(x)$ are the orthonormal polynomials

$$p_n(W^2;x) := \gamma_n(W^2)x^n + \cdots, \qquad n = 0,1,2,3,\ldots,$$

satisfying

$$\int_{-\infty}^{\infty} p_m(W^2;x)p_n(W^2;x)W^2(x)\,dx = \delta_{mn}.$$

While asymptotics for the ratio γ_{n-1}/γ_n, $n \to \infty$, are classical for weights on $[-1,1]$, only recently have analogous results been considered for the more difficult problem of weights on $\mathbf{R}$. In 1974, G. Freud [**2**] conjectured that if $W(x) = W_{\alpha,\rho}(x)$, where

$$W_{\alpha,\rho}(x) := |x|^{\rho/2}\exp(-|x|^\alpha), \qquad x \in \mathbf{R},\ \alpha > 0,\ \rho > -1,$$

then

$$\lim_{n\to\infty} n^{-1/\alpha}\gamma_{n-1}(W^2_{\alpha,\rho})/\gamma_n(W^2_{\alpha,\rho})$$

exists. He expressed the value that the limit should take in terms of gamma functions, and proved his conjecture for $\alpha = 2,4,6$. Recently, Al. Magnus [**8**] proved the conjecture for $\rho > -1$ and α a positive even integer, and subsequently [**9**] for weights of the form $\exp(-P(x))$, where $P(x)$ is a polynomial of even degree with positive leading coefficient. Máté, Nevai, and Zaslavsky [**11**] have sharpened Magnus' result to an asymptotic expansion. Several applications of Freud's Conjecture are discussed by Nevai [**16**], and related physical applications have been considered by Bessis, Itzykson, and Zuber [**1**] and in [**17**].

The purpose of this paper is to announce a proof of Freud's Conjecture for a general class of weights that includes $W_{\alpha,\rho}(x)$ for all $\alpha > 0$, $\rho > -1$. In describing the analogue of the conjecture for general weights, a crucial role is played by the number $a_n = a_n(W)$, introduced by Mhaskar and Saff in [**13**, **14**]. Let $W(x) = \exp(-Q(x))$, where $Q(x)$ is even, continuous in $\mathbf{R}$, and

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