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DIRECT IMAGES OF HERMITIAN HOLOMORPHIC BUNDLES

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Introduction. We introduce higher analogues of analytic torsion, which are form valued. Using this construction we obtain, in the case of the projection map for a product, a Grothendieck-Riemann-Roch theorem for hermitian holomorphic vector bundles which is an equality between differential forms. This is related to work of Quillen [**6**] and of Bismut and Freed [**1**].

I. A Grothendieck group.

I.1. Let X be a complex manifold. For any $p \in \mathbf{N}$ let $A^{p,p}(X)$ be the space of real (p,p) forms over X. Let $A(X) = \bigoplus_{p\geq 0} A^{p,p}(X)$, and $\tilde{A}(X) = A(X)/(\mathrm{Im}(\partial)+\mathrm{Im}(\bar\partial))$, where $d = \partial + \bar\partial$ is the standard decomposition of the exterior derivative on X.

I.2. An *hermitian holomorphic bundle* (or h.h. bundle) on X is a pair $\overline{E} = (E, h)$, consisting of a finite-dimensional complex holomorphic vector bundle E over X and a smooth hermitian scalar product h on E. Given $\overline{E}$, let ∇ be the unique connection on E which is both compatible with its complex structure and unitary for h, as in [**2**]. The closed form $\mathrm{ch}(\overline{E}) = \mathrm{Tr}(\exp((i/2\pi)\nabla^2)$ in $A(X)$ represents the Chern character of E.

I.3. Let $\hat{K}_0(X)$ be the abelian group generated by pairs $(\overline{E}, \eta)$ where E is an h.h. bundle over X and $\eta \in \tilde{A}(X)$, with the following relations. Let

$$\overline{\mathcal{E}}: 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$$

be any exact sequence of holomorphic bundles over X, endowed with arbitrary metrics, and $\eta', \eta'' \in \tilde{A}(X)$. We impose the relation $(\overline{S}; \eta') + (\overline{Q}; \eta'') = (\overline{E}; \eta' + \eta'' - \widetilde{\mathrm{ch}}(\overline{\mathcal{E}}))$, where $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \tilde{A}(X)$ is the solution to the equation

$$(1/\pi i)\partial\bar\partial\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \mathrm{ch}(\overline{S}) + \mathrm{ch}(\overline{Q}) - \mathrm{ch}(\overline{E})$$

introduced by Bott and Chern in [**2**].

I.4. The following construction of $\widetilde{\mathrm{ch}}$ is used in the proofs of the results below. Let $\mathcal{O}(1)$ be the tautological line bundle on the complex projective line $\mathbf{P}^1$, and let z be the parameter on the affine line $\mathbf{A}^1 \subset \mathbf{P}^1$. If $\sigma: \mathcal{O} \to \mathcal{O}(1)$ is the section vanishing at infinity, let $s = \mathrm{Id} \otimes \sigma$ be the induced map $S \to S(1)$ on $X \times \mathbf{P}^1$. If $i: S \to E$ is the inclusion in $\overline{\mathcal{E}}$ above, let $F = (S(1) \oplus E)/S$ be the vector bundle which is the cokernel of $s \oplus i$. If $i_p: X \times \{p\} \to X \times \mathbf{P}^1$ for $p = 0, \infty$ are the natural inclusions, then $i_0^* F \simeq E$ while $i_\infty^* F \simeq S \oplus Q$. We may choose a metric on F so that these maps are isometries. Then, in $\tilde{A}(X)$:

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \int_z \mathrm{ch}(\overline{F}) \log |z|.$$

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