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A GENERALIZATION OF THE TARSKI-SEIDENBERG THEOREM, AND SOME NONDEFINABILITY RESULTS

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This article points out some remarkable facts implicit in the results of Lojasiewicz [**L1**] and Gabrielov [**Ga**].

An important consequence of Tarski's work [**T**] on the elementary theory of the reals is a characterization of the sets which are elementarily definable from addition and multiplication on **R**. Allowing arbitrary reals as constants, this characterization consists of the identification of the definable sets with the semialgebraic sets. (A *semialgebraic subset of* $\mathbf{R}^m$ is by definition a finite union of sets of the form $\{x \in \mathbf{R}^m\colon\ p(x) = 0,\ q_1(x) > 0, \ldots, q_k(x) > 0\}$ where $p, q_1, \ldots, q_k$ are real polynomials.) The fact that the system of semialgebraic sets is closed under definability is also known as the Tarski-Seidenberg theorem, and this property, together with the topological finiteness phenomena that go with it—triangulability of semialgebraic sets [**L2, Gi**], generic triviality of semialgebraic maps [**Ha**]—make the theory of semialgebraic sets a useful analytic-topological tool.

Below we extend the system of semialgebraic sets in such a way that the Tarski-Seidenberg property, i.e., closure under definability, and the topological finiteness phenomena are preserved. The polynomial growth property of semialgebraic functions is also preserved. This extended system contains the arctangent function on **R**, *the sine function on any bounded interval, the exponential function* e^x *on any bounded interval, but not the exponential function on all of* **R**. (And it couldn't possibly contain the sine function on all of **R** without sacrificing the finiteness phenomena, and a lot more.)

As a corollary we obtain that neither the exponential function on **R**, *nor the set of integers, is definable from addition, multiplication, and the restrictions of the sine and exponential functions to bounded intervals.*

Questions of this type have puzzled logicians for a long time. (There still remain, of course, countless unsolved problems of this sort.) In a more positive spirit Tarski [**T**, p. 45] asked to extend his results so as to include, besides the algebraic operations on **R**, certain transcendental elementary functions like e^x; the theorem below is a partial answer. (More recently, Hovanskii [**Ho**, p. 562] and the author [**VdD1, VdD2**] asked similar questions, and in [**VdD3**] we

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