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MICHAEL'S PROBLEM AND THE POINCARÉ-FATOU-BIEBERBACH PHENOMENON

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1. Introduction. Let A be a Fréchet algebra, i.e. a complete, metrizable topological algebra whose topology is defined by an increasing family (q_n) of submultiplicative seminorms. It is still not known whether characters on A are necessarily continuous. This is the classical "problem of Michael" raised in 1952 by Michael in **[73]**. The starting point of this paper is a new approach to this problem. We show in Theorem 3.3 that if discontinuous characters do exist on some Fréchet algebras, then for every system $(\mathbf{C}^{p_n}, F_n)_{n \geqslant 1}$, where (p_n) is a sequence of nonnegative integers and where F_n: $\mathbf{C}^{p_{n+1}} \to \mathbf{C}^{p_n}$ is entire for every $n \geqslant 1$, there exists a sequence (z_n) in $\prod_{n \geqslant 1} \mathbf{C}^{p_n}$ such that $z_n = F_n(z_{n+1})$ $(n \geqslant 1)$. In particular, if it were possible to construct for some $p \geqslant 1$ a sequence (F_n) of entire functions from $\mathbf{C}^p$ into itself such that $\bigcap_{n \geqslant 1}(F_1 \circ \cdots \circ F_n)(\mathbf{C}^p) = \varnothing$ then the answer to Michael's problem would be positive, i.e., all characters on Fréchet algebras would be continuous. It follows of course from the big Picard theorem that no such sequence exists for $p = 1$, but if $p \geqslant 2$ it is well known that there exists a one-to-one entire function F: $\mathbf{C}^p \to \mathbf{C}^p$, whose jacobian identically equals 1 but whose range is not dense in $\mathbf{C}^p$. Such a function was constructed by Bieberbach in **[11]** for $p = 2$ (see also the very clear exposition of Bieberbach's construction presented by Stehlé in **[101]**, where the argument is extended to all $p \geqslant 2$). Another example of a nondegenerate entire function from $\mathbf{C}^2$ into itself whose range is not dense was given previously by Fatou in **[46]** (but, despite some claims of the contrary, Fatou's function is not one-to-one, see §6). In fact this phenomenon was explicitly pointed out by Poincaré in **[85**, p. 333] forty years before Bieberbach. All these constructions rely on the theory of normal forms for local analytic automorphisms with a repulsive fixed point, and suggest that if $p \geqslant 2$ a sequence (F_n) with $\bigcap_{n \geqslant 1}(F_1 \circ \cdots \circ F_n)(\mathbf{C}^p) = \varnothing$ might exist. We were not able to decide whether this is true or not, but we give here a new method to construct these "Bieberbach functions". This leads to new examples of one-to-one entire functions over $\mathbf{C}^2$, of jacobian 1, whose range is not dense, given in §8. We use these functions to construct strange nondegenerate entire functions from $\mathbf{C}^2$ into itself.

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