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ALMOST ALL p-GROUPS HAVE AUTOMORPHISM GROUP A p-GROUP

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1. Introduction. Groups of prime-power order are tantalizing objects. On one hand they have a delicate and sophisticated combinatorial structure related to representations of $\mathrm{GL}(n,p)$ in characteristic p; on the other there are so many of them and their structure is so varied that any kind of classification seems hopeless and powerful general theorems are rare.

This paper is concerned with the proof that a random group of prime-power order has no automorphisms of order coprime to p. Although in the course of the proof we establish several new combinatorial results about finite p-groups, the result gives further evidence of the apparent structurelessness of groups of prime-power order. The result may not seem entirely plausible at first sight, since most groups of prime-power order with which we are familiar arise as subgroups of Chevalley groups or simple groups and admit automorphisms of order coprime to p. Indeed, apart from the dihedral group of order eight, the known examples of such groups are given by complicated and unnatural-looking constructions. Intuitively, what our result is saying is that most p-groups are complicated and unnatural-looking, and that the familiar examples are far from typical.

At the heart of our proof lies the combinatorics which links finite p-groups and representations of the general linear group $\mathrm{GL}(n,p)$. Isomorphism classes of abelian groups of order p^n correspond to partitions of n and Hall [**Ha**] showed that questions about these groups could be answered in terms of polynomials which form an algebra which can be identified with the algebra of symmetric functions. An account is given in [**Mac**]. Hall's polynomials can be used to give exact formulae for the number of subgroups of an abelian p-group and were used by Green [**Gr**] to determine the characters of the general linear group. In this work we extend Hall's results to give upper bounds for the number of normal subgroups of a nonabelian p-group, and use similar techniques to estimate the number of subspaces of a $\mathrm{GL}(n,p)$ module in characteristic p which are left invariant by no non-identity element of the group.

2. Statement of results. To state our results precisely we introduce some notation. For any prime p and group H the Frattini series is defined as

$$H_1 = H$$

and

$$H_{i+1} = H_i^p[H, H_i] \quad \text{for } i \geq 1.$$

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