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THE MODULI SPACE OF A PUNCTURED SURFACE AND PERTURBATIVE SERIES

R. C. PENNER

0. Introduction. Let F_g^s denote the oriented genus g surface with s punctures, $2g-2+s>0$, $s\geq 1$, and choose a distinguished puncture P of F_g^s. Let $\mathcal{T}_g^s$ be the *Teichmüller space* of conformal classes of complete finite-area metrics on F_g^s (see [**A**]), and let MC_g^s denote the *mapping class group* of orientation-preserving diffeomorphisms of F_g^s (fixing P) modulo isotopy (see [**B**]). When g,s are understood, we omit their mention. In §1 and §2, we report on joint work with D. B. A. Epstein [**EP**] where new and useful coordinates on $\mathcal{T}_g^s$ are given (Theorem 2) and a MC_g^s-equivariant cell decomposition of $\mathcal{T}_g^s$ is described (Theorem 3). There is thus an induced cell decomposition of the quotient $\mathcal{M}_g^s=\mathcal{T}_g^s/MC_g^s$, which is the usual *moduli space* of F_g^s in case $s=1$. In §3, we describe a remarkable connection (see [**P**]) between this cell-decomposition for $s=1$ and a technique from quantum field theory, which allows the computation of certain numerical invariants of $\mathcal{M}_g^s$ (Corollary 6). Analogues of Theorem 3 have been obtained independently by [**BE** and **H**] using different techniques. Furthermore, Corollary 7 is in agreement with some recent work in [**HZ**].

Let M denote Minkowskii 3-space with bilinear pairing $\langle\cdot,\cdot\rangle$ of type $(+,+,-)$, and let $L^+\subset M$ denote the (open) positive light-cone. The uniformization theorem (see [**A**]) allows us to identify $\mathcal{T}_g^s$ with the space of (conjugacy classes of faithful and discrete) representations of $\pi_1(F_g^s)$ in $SO(2,1)$ (as a Fuchsian group of the first kind in the component of the identity).

I. Coordinates on $\mathcal{T}$. Suppose $\pi_1 F=\Gamma\in\mathcal{T}$, and choose a parabolic transformation $\gamma\in\Gamma$ corresponding to the puncture P. γ fixes a unique ray in L^+, and we choose a point $z\in L^+$ in this ray. If c is a bi-infinite geodesic in F which tends in both directions to P (to be termed simply a *geodesic* in the sequel), let $\gamma(c)\in\Gamma$ denote the corresponding translation, and define the λ-*length* of (the homotopy class of) c to be $\lambda_\Gamma(c)=\sqrt{-\langle z,\gamma(c)z\rangle}$. When Γ is understood, we denote $\lambda(c)=\lambda_\Gamma(c)$. If h is a Γ-horosphere about P and c is a Γ-geodesic, then we define $d_h(c)$ to be the Γ-hyperbolic length along c from h back to h.

LEMMA 1. *If c_1 and c_2 are geodesics, then*

$$\lim_{h\to P}\exp\{d_h(c_1)-d_h(c_2)\}=[\lambda(c_1)/\lambda(c_2)]^2.$$

It follows that λ-lengths are natural in the sense that if $\varphi\in MC$, $\Gamma\in\mathcal{T}$, and c_1,c_2 are geodesics, then $\lambda_{\varphi^*\Gamma}(c_1)/\lambda_{\varphi^*\Gamma}(c_2)=\lambda_\Gamma(\varphi^{-1}c_1)/\lambda_\Gamma(\varphi^{-1}c_2)$,

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