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DYNAMICS OVER TEICHMÜLLER SPACE

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1. Introduction. Let $p, n \geq 0$ be such that $3p - 3 + n > 0$, and let $\mathcal{T}_{p,n}$ be the Teichmüller space of marked closed Riemann surfaces of genus p with n punctures. (See [**1**] for definitions and references.) Equipped with the Teichmüller metric, $\mathcal{T}_{p,n}$ is a straight line space: through each pair of distinct points there passes a unique isometric copy of $\mathbf{R}$ [**3**]. It is also known these "Teichmüller geodesics" are the projections in $\mathcal{T}_{p,n}$ of orbits of a geodesic flow, the geodesic flow on the unit cotangent bundle, $\mathcal{Q}^1_{p,n}$, relative to a Finsler metric [**9**].

If $r\colon \mathcal{Q}_{p,n} \to \mathcal{T}_{p,n}$ is the cotangent bundle projection, each $q \in \mathcal{Q}_{p,n}$ may be identified as a holomorphic quadratic differential of finite norm on any surface in the class of $r(q) \in \mathcal{T}_{p,n}$. Moreover, this differential extends to be meromorphic on the closed surface with at worst simple poles at the punctures. While the choice of surface is not canonical, the singularity pattern of the differential is. That is, we may attach to q a "divisor" $\sigma = \sigma(q) = (k, \nu, \varepsilon)$, where k is the number of poles, $\nu(j)$, $j \geq 1$, is the number of zeros of order j, and $\varepsilon = +1$ or -1 as (any representative of) q is or is not the square of a 1-form.

Given σ as in the previous paragraph, we consider the set $\mathcal{Q}_{p,n}(\sigma)$ of $q' \in \mathcal{Q}_{p,n}$ which have $\sigma(q') = \sigma$. With obvious notation it is a consequence of the Teichmüller theorem that $\mathcal{Q}^1_{p,n}(\sigma)$ is invariant under the geodesic flow. Let $\Gamma = \Gamma(p,n)$ be the mapping class group (modular group) of a closed oriented surface of genus p with n punctures. Γ acts canonically on $\mathcal{Q}^1_{p,n}(\sigma)$, and $\mathcal{Q}^*_{p,n}(\sigma)$ denotes the "moduli space", $\mathcal{Q}^*_{p,n}(\sigma) = \mathcal{Q}^1_{p,n}(\sigma)/\Gamma$. The geodesic flow commutes with Γ and projects to a flow on $\mathcal{Q}^*_{p,n}(\sigma)$. We shall describe briefly the results of [**11–13**]: *On each component of $\mathcal{Q}^*_{p,n}(\sigma)$ the geodesic flow is "measurably Anosov" with metric entropy a simple function of σ.* A closing lemma yields a lower bound for the growth of the "Teichmüller length spectrum".

2. *F*-structures. It is possible to realize the spaces $\mathcal{T}_{p,n}$ and $\mathcal{Q}_{p,n}$ as spaces of classes of atlases. To be more precise, let $M_{p,n}$ be a fixed closed oriented surface of genus p with n punctures, the puncture set denoted by $S_n = \{s_j \mid 1 \leq j \leq n\}$. $H(p,n)$ (resp. $H_0(p,n)$) denotes the group of orientation-preserving homeomorphisms φ of M_p such that $\varphi S_n = S_n$ (resp. such that $\varphi \sim \mathrm{Id}(\mathrm{rel}\, S_n)$). Of course, $\Gamma = H(p,n)/H_0(p,n)$. Let $\mathcal{C}_p$ be the set of maximal

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