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THE FIRST EIGENVALUE IN A TOWER OF COVERINGS

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Let M be a compact Riemannian manifold, and let $\{M_i\}$ be a family of finite Riemannian covering spaces of M. Let $\lambda_1(M_i)$ be the first eigenvalue of the Laplacian on M_i. λ_1 is given by the variational formula

$$\lambda_1(M_i) = \inf_f \frac{\int_{M_i} ||df||^2}{\int_{M_i} |f|^2},$$

where f ranges over functions satisfying $\int_{M_i} f = 0$.

In this note we announce results on the following problem: When is there a sequence of i's where $\lambda_1(M_i)$ is bounded from below as $i \to \infty$? Our approach to this problem is of a piece with our approach to studying eigenvalue problems related to λ_0 in [**1** and **3**]. Namely, we reduce the eigenvalue problem to a combinatorial problem built out of the fundamental group.

To state the combinatorial problem, let us pick generators $g_1, \ldots, g_k$ for $\pi_1(M)$. Consider, for each i, the finite graph Γ_i described as follows: the vertices of Γ_i are the cosets $\pi_1(M)/\pi_1(M_i)$. Two vertices are joined by an edge if they differ by left multiplication by one of the g_i's.

For each i, let $h_i = h(\Gamma_i)$ denote the following number: Let $E = \{E_j\}$ be a collection of edges of Γ_i such that $\Gamma_i - E$ disconnects into two pieces,

$$\Gamma_i - E = A \cup B.$$

Denote by $\#(E)$ the number of elements of E, and $\#(A)$ (resp. $\#(B)$) the number of vertices in A (resp. B). Then

$$h_i = \inf_E \frac{\#(E)}{\min(\#(A), \#(B))}.$$

h_i is of course the combinatorial analogue of Cheeger's isoperimetric constant [**11**].

THEOREM 1. *There is a positive constant C_1 such that $\lambda_1(M_i) \geq C_1$ for all i if and only if there is a positive constant C_2 such that $h_i \geq C_2$ for all i.*

The general question of describing manifolds M for which $\lambda_1(M_i)$ is uniformly bounded away from 0 was raised recently by Sunada [**10**]. Letting $\{M_i\}$ range over all coverings of M, Theorem 1 provides a combinatorial answer to his question. Of course, the problem remains of finding good criteria to check when this combinatorial problem is solved. We comment on some examples at the end of this paper.

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