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AMALGAMS OF L^p AND l^q

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1. Introduction. The *amalgam* of L^p and l^q on the real line is the space (L^p, l^q) consisting of functions which are locally in L^p and have l^q behavior at infinity in the sense that the L^p-norms over the intervals $[n, n+1]$ form an l^q-sequence. For $1 \leqslant p, q < \infty$, the norm

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right\}^{1/q} \tag{1.1}$$

makes (L^p, l^q) into a Banach space.

The idea of considering the amalgam (L^p, l^q), as opposed to the Lebesgue space $L^p = (L^p, l^p)$, is a natural one because it allows us to separate the global behavior from the local behavior of a function. This idea goes back to 1926 and Norbert Wiener who considered the special cases (L^1, l^2) and (L^2, l^∞) in **[W1]** and (L^∞, l^1) and (L^1, l^∞) in **[W2]**. Other special cases have appeared sporadically since then, but the first systematic study of these spaces was undertaken in 1975 by F. Holland **[H1]**.

After giving an account of the basic theory of amalgams on groups in §2, we show in the following sections how amalgams have arisen in various areas of analysis: almost periodic functions **[W1]**, Tauberian theorems **[W2]**, extending the domain of the Fourier transform **[Sz1]**, Fourier multipliers **[EHR]**, integral operators **[BiS]**, product-convolution operators **[BuS]**, positive definite functions **[Coop]**, Fourier transforms of unbounded measures **[H2]**, lacunarity **[Fou2]**, the lower majorant property for $H^p(R^n)$ **[BaS]**, approximation theory **[JR]**, algebras and modules **[LVW]**, and the range of the Fourier transform **[Ke]**. The common theme is that, in many situations, an amalgam space (L^p, l^q) turns out to be exactly the right space that is needed to solve a problem or formulate a theory.

In view of these occurrences, "the amalgam spaces ... appear to be an idea whose time has come" **[GdL]**. We hope that the present article will help to make these spaces more widely known in the mathematical community.

2. Amalgams on groups. If G is a locally compact abelian group, we use the structure theorem to write $G = R^a \times G_1$, where a is a nonnegative integer and G_1 is a group with a compact open subgroup H. We let $I = [0,1)^a \times H$ and

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