BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 2, April 1985

0273-0979/85 $1.00 + $.25 per page

Amarts and set function processes, by Allen Gut and Klaus D. Schmidt, Lecture Notes in Mathematics, vol. 1042, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 258 pp., $12.50. ISBN 3-540-12867-0

What is an amart? Let (X_n) be a sequence of random variables adapted to increasing sigma algebras $\mathscr{L}_n$. A stopping time is a random variable T taking positive integer values and the value $+\infty$ such that if $n < \infty$, then the event $\{T = n\}$ (equivalently, $\{T \leqslant n\}$) is in $\mathscr{L}_n$. Intuitively, this means that the event T equals n is determined by the outcome of the trials up to the time n. If X_n is the fortune of a gambler at time n and the casino gives no credit, then the time when a ruined gambler must stop is also a stopping time in the mathematical sense. Stopping times, as propounded by J. L. Doob and, later, by the Strasbourg school led by P. A. Meyer, are among the most important features of modern probability. For convergence problems, of special importance are *simple* stopping times: those taking finitely many finite values. (Simple stopping times are also the only ones of practical importance: They do not require an infinite amount of time or an infinite fortune.) Let Σ be the collection of simple stopping times. A martingale can be defined by the property: the net $(EX_T$: T in $\Sigma)$ is constant. If the same net *converges*, the process (X_n) is called an *amart* (originally an acronym for asymptotic martingale). Since amarts are to include martingales, it is crucial in this definition to allow only *simple* stopping times: Otherwise a martingale need not be an amart, as seen by the example of the famous original gambling martingale in which the player doubles his stake each time he loses. After the first article by John Baxter [5], the amart convergence theorem—asserting almost sure convergence of L_1-bounded amarts—was proved by D. G. Austin, G. A. Edgar, and A. Ionescu Tulcea (= A. Bellow) [3]. The same result was obtained in a less explicit but stronger form by R. V. Chacon [12]. Chacon's "Fatou's inequalities" were anticipated by W. D. Sudderth [40], who was influenced by L. Dubins, but Sudderth considered nonsimple stopping times. The amart convergence theorem, and more, was proved earlier in a measure-theoretical form involving *no* stopping times by C. Lamb [29]. We submit, however, that the amart theory started with simple stopping times, and that, in fact, the notion of simple stopping time is more basic here than the exact definition of the amart, since the deeper convergence theorems on directed sets have various forms, all of which share the use of simple stopping times. More about this later. The article of Chacon and the reviewer [13] initiates vector-valued amarts. These authors are the first to believe that the notion merits a name: *asymptotic martingale*. A systematic presentation of the amart theory paralleling the martingale theory, including for the first time the optional sampling theorem, the Riesz decomposition, the descending and the continuous parameter cases, was given by Edgar and the reviewer [21]. This article calls a spade a spade, introducing the