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*Semigroups of linear operators and applications to partial differential equations*, by A. Pazy, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983, viii + 279 pp., $29.80. ISBN 0-387-90845-5

If $f$ is a continuous, real-valued function on $[0, \infty)$ that satisfies $f(0) = 1$ and the semigroup property $f(t + s) = f(t)f(s)$ for $t, s \geqslant 0$, then it is easy to see that there is a real number $a$ such that $f(t) = e^{ta}$ for $t \geqslant 0$ [one can, for example, note that $t \to \ln(f(t))$ is continuous and additive on $[0, \infty)$, and so $\ln(f(t)) = ta$ for $t \geqslant 0$, where $a = \ln(f(1))$]. The number $a$ can also be computed directly from $f$ by the formula

$$(1) \qquad a = \lim_{h \to 0^+} \frac{f(h) - 1}{h} = \frac{d^+}{dt} f(t) \Big]_{t=0}.$$

This observation is important because it connects $f$ to the initial-value problem

$$(2) \qquad u'(t) = au(t), \quad t \geqslant 0, \qquad u(0) = z.$$

In particular, $u(t) \equiv f(t)z$ is the solution to (2). Conversely, given the initial-value problem (2), there are many ways to construct the solution directly: The three formulas

$$(3) \quad e^{ta} = \sum_{n=0}^{\infty} \frac{t^n}{n!} a^n, \quad e^{ta} = \lim_{n \to \infty} \left(1 + \frac{t}{n} a\right)^n, \quad e^{ta} = \lim_{n \to \infty} \left(1 - \frac{t}{n} a\right)^{-n}$$

are known from elementary calculus to be valid.

Given a basic background in functional analysis, this elementary discussion can be readily extended to various types of initial-value problems. Suppose that $H$ is a Hilbert space with inner product denoted by $\langle \cdot , \cdot \rangle$ and that $\{e_n\}_1^\infty$ is a complete orthonormal sequence in $H$. Suppose further that $\{\lambda_n\}_1^\infty$ is a decreasing sequence of real numbers, and for each $t \geqslant 0$ and $z \in H$ define

$$(4) \qquad S(t)z = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z, e_n \rangle e_n.$$

It is easy to see that $S(t)$ is a bounded linear operator on $H$, with $\|S(t)\| \leqslant e^{\lambda_1 t}$ for all $t \geqslant 0$, and the orthogonality of $\{e_n\}_1^\infty$ implies that the semigroup property $S(t)S(s)z = S(t + s)z$ holds for all $t, s \geqslant 0$ and $z \in H$. A formal calculation shows that

$$\frac{d}{dt} S(t)z \Big]_{t=0} = \sum_{n=1}^{\infty} \lambda_n \langle z, e_n \rangle e_n,$$