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A THRESHOLD FOR A CARICATURE OF THE NERVE EQUATION

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1. Introduction. Hodgkin-Huxley [**1952**][2] described the conduction of the nervous impulse in the optical nerve of a squid.[3] The physiological fact to be modelled is that stimuli *below* threshold damp out and so convey no information, but a stimulus *above* threshold is rapidly converted into a train of pulses of approximately fixed shape and spacing that travel down the line with little distortion. Fitzhugh [**1961/69**] and Nagumo et al. [**1964**] proposed a simplified model:[4]

$$(1)\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) + e^{-ct}f + be^{-ct} * u, \qquad (t,x) \in [0,\infty] \times \mathbf{R},$$

in which $0 < a < 1$, $b \le 0$ is a (small) coupling, $c > 0$ is a damping, and f is (part of the) initial data. McKean [**1970**] suggested replacing the cubic in (1) by a broken line of the same general shape. The present announcement deals with this caricature:

$$(2)\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + (1 \text{ if } u > a) + e^{-ct}f + be^{-ct} * u, \qquad (t,x) \in [0,\infty] \times \mathbf{R}.$$

It is believed, but not proven, that (1) and (2) have similar portraits in the large. This would be helpful since (2) is much more tractable than (1). The mathematical problem posed by (2) (or (1)) is to classify its waveforms and to prove that every solution of a suitable initial size and shape, tracked at the proper speed, converges to one of them. The actual diversity of waveforms for (2) is quite staggering, as Feroe [**1981**] and others have confirmed: For suitable values of a, b, c, (2) admits trains of $1, 2, 3, \ldots, \infty$ pulses; moreover, the ∞-pulse trains come in many varieties. In view of this complexity it is natural to classify the waveforms according to the number of crossings of the level a and to begin with the simplest cases.

2. One crossing: rising waves. McKean [**1983/84**] dealt with this case for small $b, c > 0$ and $a(1 - b/c) < 1$; the same result was obtained for $b = 0$ by Fife-Mcleod [**1977**] for (1) and by Terman [**1983**] for (2). The fact is that, up to translation, (2) has a single waveform $w(x)$ which rises from

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[2]Hodgkin [**1971**]

[3]Cohen [**1976**], Hadeler [**1976**], and Rinzel [**1976**] review the model and its simplifications.

[4]$*$ means *convolution.*