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A COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM ASSOCIATED WITH LINE FITTING IN COMPLEX VECTOR SPACES

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Introduction. Over the past decade there has been a great deal of work on the explicit integration of completely integrable Hamiltonian systems. (See Adler and Van Moerbeke [**1**], McKean [**9**], Moser [**11**], Mumford [**13**].) Among the systems that have been studied are the free n-dimensional rigid body, the Euler-Poisson equations, geodesic flow on an ellipsoid, Neumann's equations, the Toda lattice, and Nahm's equations. The flows of these systems can be shown to linearize on the real part of the Jacobi variety of an algebraic curve associated with the system (see Adler and Van Moerbeke [**1**], Griffiths [**7**]).

Now, all of the above systems, except Nahm's equations, which arise in the theory of monopoles, come from problems in classical mechanics.

Here we present and explicitly integrate a completely interable Hamiltonian system that arises in a statistical problem—the fitting of lines to a data set in a complex vector space.

Remarkably, this system fits into the general scheme for integrating systems of "spinning top and ellipsoid type" developed by Moser [**11**] and Adler and Van Moerbeke [**1**]. Further, both the considered integrals and the flow have an interesting statistical meaning.

1. Let $\mathbf{C}^n$ denote complex n-dimensional euclidean space with orthonormal basis e_i, $i = 1, \ldots, n$. Let $x_i = \sum_{j=1}^{n} \lambda_{ij} e_j$, $i = 1, \ldots, p$ be p data points. Then the total least squares estimate (see Golub and Van Loan [**6**]) of a d-plane fitted to the data set is given by minimizing the total perpendicular distance of the points from the plane. (In the case of lines this corresponds to the first principal component of the data (see Kendall [**8**]).)

The distance function is given by

$$H(Q) = \operatorname{Tr} C(I - Q) = \operatorname{Tr} CP,$$

where $P = I - Q$, $Q =$ orthogonal projection matrix of $\mathbf{C}^n$ onto the d-plane, and C is a matrix with entries $c_{kj} = \sum_i \lambda_{\overline{ij}} \lambda_{ik}$.

Now let $G_{\mathbf{C}}(d,n) =$ the complex Grassmanian of d-planes in n-space, $u(n) =$ the Lie algebra of the unitary group $U(n)$, $h(n) = n \times n$ Hermitian matrices. Note that $P, C \in h(n)$, P having rank $n - d$. Remarkably then, H is the restriction of a linear functional to $C_{\mathbf{C}}(d,n)$, viewed as an adjoint orbit of $u(n)$ (with rank $n - d$ matrices). Further, since $U(n)$ is compact, adjoit and coadjoint orbits may be identified via the Killing form. Then H may be regarded as Hamiltonian on an adjoint orbit of $u(n)$ with the inherited Kostant-Kirilov symplectic structure.

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