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NONLINEAR STABILITY OF SHOCK WAVES FOR VISCOUS CONSERVATION LAWS

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Consider the viscous conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=\frac{\partial}{\partial x}\left(B(u)\frac{\partial u}{\partial x}\right), \qquad t\geq 0,\ -\infty<x<\infty, \tag{1}$$

where $u=u(x,t)\in \mathbf{R}^n$, the flux $f(u)$ is a smooth n-vector-valued function, and the viscosity $B(u)$ is a smooth $n\times n$ matrix. We are interested in the stability of traveling waves, the "viscous shock waves", for (1). It is shown that when the initial data are a perturbation of viscous shock waves, then the solution converges to these viscous shock waves, properly translated in space, in the uniform sup norm as time t tends to infinity. Our analysis is based on the observation that a general perturbation also gives rise to diffusion waves in addition to translating viscous shock waves. A new technique combining the characteristic method and the energy method is introduced for the stability analysis. The energy method is a standard technique for parabolic systems. We use the method of characteristics, usually associated with hyperbolic systems, because, physically, the viscious shock waves and nonlinear diffusion waves are nonlinear hyperbolic waves in some general sense. This characteristic-energy method is based on a new understanding of nonlinear diffusion waves and, in particular, on their characterization as compression waves and weak expansion waves.

We assume that the associated hyperbolic conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \qquad u\in \mathbf{R}^n, \tag{2}$$

are strictly hyperbolic; that is, $\partial f(u)/\partial u$ has real and distinct eigenvalues $\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u)$:

$$\frac{\partial f(u)}{\partial u}r_i(u)=\lambda_i(u)r_i(u),$$

$$l_i(u)\frac{\partial f(n)}{\partial u}=\lambda_i(u)l_i(u), \qquad i=1,2,\ldots,n.$$

We assume that each characteristic field is either genuinely nonlinear or linearly degenerate [**5**]. The behavior of shock waves, N-waves, and linear waves for (2) is well understood. It has been shown that a perturbation of shock waves gives rise to N-waves and linear waves. N-waves are described qualitatively by N-waves for the scalar equation

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