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A SYMPLECTIC FIXED POINT THEOREM FOR COMPLEX PROJECTIVE SPACES

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1. Arnold's conjecture. An automorphism ψ of a symplectic manifold (P, ω) is *homologous to the identity* if there is a smooth family ψ_t ($t \in [0,1]$) of automorphisms such that the time-dependent vector field ξ_t defined by $d\psi_t/dt = \xi_t \circ \psi_t$ is globally hamiltonian; i.e. if there is a smooth family H_t of real-valued functions on P such that $\xi_t \rfloor \omega = dH_t$. It was conjectured by Arnold [**1**], as an extension of the Poincaré-Birkhoff annulus theorem [**3**, **7**], that every automorphism of a compact symplectic manifold P, homologous to the identity, has at least as many fixed points as a function on P has critical points.

Arnold's conjecture was proven by Conley and Zehnder [**4**] for the torus $T^{2n} \approx \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ with its usual symplectic structure. They show that every symplectic automorphism of T^{2n}, homologous to the identity, has at least $n+1$ fixed points, and at least 2^{2n} if all are nondegenerate. Their method was extended in [**8**] to prove a version of Arnold's conjecture for arbitrary P under the additional assumption that the hamiltonian vector field ξ_t is sufficiently C^0 small.

In this note we announce a proof of Arnold's conjecture for the complex projective space $\mathbf{C}P^n$ with its standard symplectic structure. We prove that a symplectic diffeomorphism of $\mathbf{C}P^n$, homologous to the identity, has at least $n+1$ *distinct* fixed points. (By the Lefschetz fixed point theorem, any continuous map from $\mathbf{C}P^n$ to itself, homotopic to the identity, has at least $n+1$ fixed points *counted with multiplicities.*) For $n = 1$ ($\mathbf{C}P^1 \approx S^2$) the result was already known [**1**], but with a proof which worked only in this two-dimensional case.

The proof for T^{2n} in [**4**] made use of a variational principle in which the fixed points of the map were identified with periodic solutions of a time-dependent hamiltonian system and then identified with critical points of a functional on the space of contractible loops on T^{2n}. The corresponding functional in the case of $\mathbf{C}P^n$ is multiple valued, and there are other difficulties connected with the curved geometry of $\mathbf{C}P^n$, so we need a new approach. Our trick is to consider the hamiltonian system on $\mathbf{C}P^n$ as the *reduction*, in the sense of [**6**], of a hamiltonian system on $\mathbf{C}^{n+1}$ and then adapt recently developed methods [**2**] for finding periodic orbits in $\mathbf{C}^{n+1}$. This method is similar to that of Conley and Zehnder in that a problem on a compact manifold is lifted to a problem on euclidean space invariant under a group of transformations.

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