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C^*-ALGEBRAS AND DIFFERENTIAL TOPOLOGY

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Let M be a smooth closed manifold. If D is an elliptic differential operator on M, then the differential structure on M is explicitly involved in the definition of the analytic index of D. It is a consequence of the Atiyah-Singer Index Theorem that this integer only depends on the homeomorphism type of the manifold M, since the topological formula for the index involves the rational Pontrjagin classes which are topological invariants.

By considering families of operators one may determine a more refined index for an elliptic operator which will lie in $K_0(M)$ [**1**]. This raises the possibility of torsion (i.e., finite order) invariants for operators. We exploit this to study the dependence of the algebra of 0th-order pseudodifferential operators on the underlying differential structure.

The BDF theory of C^*-algebra extensions [**2**] provides a formalism for studying such questions. Recall that the algebra of 0th-order pseudodifferential operators on a manifold $\mathcal{P}_0$ defines an extension of C^*-algebras $0 \to \mathcal{K} \to \mathcal{P}_0 \to C(SM) \to 0$, where SM is the tangent sphere bundle of M. We denote this by $\mathcal{P}_M \in \operatorname{Ext}(SM)$. There is a natural isomorphism $\Gamma\colon \operatorname{Ext}(SM) \to K_1(SM)$. Since SM is a Spin^c manifold, there is a topologically defined K-theory fundamental class $[SM] \in K_1(SM)$.

THEOREM 1. *The map* $\Gamma\colon \operatorname{Ext}(SM) \to K_1(SM)$ *satisfies* $\Gamma(\mathcal{P}_M) = [SM]$.

This follows from the index theorem for families of operators [**5**].

We now study the question of whether $\mathcal{P}_M$ depends on the smooth structure on M. Recall that the isotopy classes of smooth structures on M can be made into a finite abelian group $\mathcal{S}(M)$. We denote by M_α the manifold M with the differential structure $\alpha \in \mathcal{S}(M)$. The identity map $1\colon M_\alpha \to M$ induces a map $\bar{1}\colon SM_\alpha \to SM$. There is a unit, $u \in K^0(SM)$, such that $\bar{1}_*([SM_\alpha]) = u \cap [SM]$. Further, there is a unit $\theta(\alpha) \in K^0(M)$, depending only on the class of $\alpha \in \mathcal{S}(M)$, which is a lift of u in the sense that $\pi^*(\theta(\alpha)) = u$, where $\pi\colon SM \to M$ is the projection.

Thus, θ defines a map from $\mathcal{S}(M)$ to $K^0(M)$.

THEOREM 2 [**5**]. *The function* $\theta\colon \mathcal{S}(M) \to K^0(M)$ *is a homomorphism of* $\mathcal{S}(M)$ *into the multiplicative group of units* $1 \oplus \tilde{K}^0(M) \subseteq K^0(M)$.

The next step is to interpret θ homotopy theoretically. Here one must work separately on the 2-primary and odd-primary parts of $\mathcal{S}(M) = \mathcal{S}(M)_{(2)} \oplus \mathcal{S}(M)_{(\mathrm{odd})}$. The two analyses proceed in a parallel way, so we sketch only

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