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WEIGHTED NORM INEQUALITIES FOR POTENTIALS WITH APPLICATIONS TO SCHRÖDINGER OPERATORS, FOURIER TRANSFORMS, AND CARLESON MEASURES

BY R. KERMAN[1] AND E. SAWYER[2]

I. Introduction. A new characterization of the trace inequality for potential operators is given and used to sharpen recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators. It is also used to study the domain and essential spectrum of Schrödinger operators, to obtain weighted norm inequalities for Fourier transforms, and to determine the Carleson measures for Dirichlet-type spaces.

THEOREM 1. *Suppose K is a nonnegative, locally integrable, radial function on R^n, which is decreasing as a function of $|x|$. For f in the class $P(R^n)$ of nonnegative, measurable functions on R^n and $x \in R^n$, set*

$$(Tf)(x) = (K * f)(x) = \int_{R^n} K(x-y)f(y)\,dy,$$

provided this integral exists for almost all $x \in R^n$. Then given $1 < p < \infty$ and $v \in P(R^n)$, there exists $C > 0$ so that the trace inequality

$$(1) \qquad \int_{R^n} (Tf)(x)^p v(x)\,dx \le C \int_{R^n} f(x)^p\,dx, \qquad f \in P(R^n),$$

holds if and only if $C' > 0$ exists with

$$(2) \qquad \int_Q T(\chi_Q v)(x)^{p'}\,dx \le C' \int_Q v(x)\,dx < \infty \quad \text{for all dyadic cubes } Q,$$

where, as usual, $p' = p/(p-1)$.

Alternative characterizations of the trace inequality in terms of L^p capacities have been obtained in [**1** and **4**].

The trace inequality (1), for $p = 2$, and the potential kernel $K^\alpha(x)$, with $\hat{K}^\alpha(\varsigma) = (\alpha + |\varsigma|^2)^{-1/2}$, arises in estimating the eigenvalues[3] of a Schrödinger operator H. Let

$$(I_2 f)(x) = \int_{R^n} |x-y|^{2-n} f(y)\,dy$$

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[3]By eigenvalues we mean the numbers $\lambda_1 < \cdots < \lambda_N < \cdots$, where λ_N is the maximum over all $N-1$ tuples $\phi_1, \ldots, \phi_{N-1}$ of the quantity $\inf\langle Hu, u\rangle/\langle u, u\rangle$, the infimum being over all $u \in Q(H)$, $u \perp \phi_j$, $j = 1, \ldots, N-1$. Here $Q(H)$ denotes the form domain of H. See [**10**].