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CHARACTERISTIC CLASSES OF SURFACE BUNDLES

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In this paper we define characteristic classes of surface bundles, namely smooth fibre bundles whose fibres are a closed orientable surface Σ_g of genus $g \geq 2$, and announce some nontriviality results for them. As a consequence we obtain lower bounds for the Betti numbers of the mapping class group $M(g)$ of Σ_g.

It is known [**EE**] that the connected component of the identity of $\mathrm{Diff}_+ \Sigma_g$, the group of orientation preserving diffeomorphisms of Σ_g, is contractible. Therefore $B\mathrm{Diff}_+ \Sigma_g$ is a $K(M(g), 1)$. Now let ξ be the tangent bundle along the fibres of an *oriented* surface bundle and let $e(\xi)$ be its Euler class. If we apply the Gysin homomorphsm to $e^{i+1}(\xi)$, we obtain an integral cohomology class of the base space of degree $2i$. By naturality this defines certain cohomology classes $e_i \in H^{2i}(M(g)\colon \mathbf{Z})$ $(i = 1, 2, \ldots)$. $M(g)$ acts on $H^1(\Sigma_g; \mathbf{Z})$, preserving the symplectic form given by the cup product, so we obtain a homomorphism $M(g) \to \mathrm{Sp}(2g; \mathbf{Z})$, where $\mathrm{Sp}(2g; \mathbf{Z})$ is the group of all $2g \times 2g$ symplectic matrices with integral entries. This induces a homomorphism $M(g) \to \mathrm{Sp}(2g; \mathbf{R})$. Since $\mathrm{Sp}(2g; \mathbf{R})$ has $\mathrm{U}(g)$ as a maximal compact subgroup, we have a g-dimensional complex vector bundle η on $K(M(g), 1)$. Let $c_i(\eta) \in H^{2i}(M(g); \mathbf{Z})$ be its ith Chern class. From the argument of Atiyah in [**A**] and the fact that η is flat as a real vector bundle, we can conclude

$$e_{2i-1} = (-1)^i (2i/B_i) s_{2i-1}(c(\eta)) \qquad (i = 1, 2, \ldots \text{and coefficients are in } \mathbf{Q}),$$
$$s_{2i}(c(\eta)) = 0$$

where $s_i(c(\eta))$ stands for the characteristic class of η corresponding to the formal sum $\sum_j t_j^i$, and B_i is the ith Bernoulli number. These two relations induce those among monomials of e_{2i-1}'s and the quotient

$$\mathbf{Q}[e_1, e_3, \ldots]/(\text{relations})$$

is naturally isomorphic to the relative Lie algebra cohomology $H^*(\mathfrak{sp}(2g; \mathbf{R}), \mathfrak{u}(g))$, which in turn is *additively* isomorphic to $H^*(S^2 \times S^4 \times \cdots \times S^{2g}; \mathbf{Q})$ (see [**BH**]). It is known that $M(g)$ acts properly discontinuously on the Teichmüller space $T(g) \cong \mathbf{R}^{6g-6}$ with noncompact quotient $\mathcal{M}_g$, the moduli space for Riemann surfaces of genus g. Hence $\mathrm{vcd}(M(g)) \leq 6g - 7$. Thus any monomial of e_i's of degree $\geq 6g - 6$ vanishes. To sum up we have a homomorphism

$$\phi\colon \mathbf{Q}[e_1, e_2, \ldots]/(\text{above relations}) \to H^*(M(g); \mathbf{Q});$$

here we use the letters e_i for both symbolic and actual meanings. Since $\mathrm{vcd}(M(g))$ is conjectured to be $3g - 3$ [**Hv**], ϕ will surely still have a large kernel. Our main results are

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