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RANDOM WALK ON THE SPEISER GRAPH OF A RIEMANN SURFACE

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ABSTRACT. We consider the problem of determining the conformal type—hyperbolic or parabolic—of a covering surface of the Riemann sphere with n punctures. To such a surface there corresponds a graph called the Speiser graph of the covering, and it is natural to ask for a criterion for the type of the surface in terms of properties of the graph. We show how to define a random walk on the vertices of the graph, so that the random walk is transient if and only if the surface is hyperbolic.

1. The type problem. A simply-connected open Riemann surface is conformally equivalent either to the open unit disk or to the entire complex plane [**1**]. In the first case the surface is said to be *hyperbolic*, or to have hyperbolic type; in the second case it is said to be *parabolic*. This dichotomy is extended to multiply-connected surfaces by declaring a surface to be hyperbolic if, like the unit disk, it has finite electrical resistance out to infinity, and parabolic if, like the plane, it has infinite resistance. Equivalently, a hyperbolic surface is one on which Brownian motion is transient, and a parabolic surface is one on which Brownian motion is recurrent [**8, 9**]. The *classical type problem* for Riemann surfaces is the problem of determining whether a given open Riemann surface is hyperbolic or parabolic.

2. The Speiser graph of a covering surface. One special case of the type problem that has received a lot of attention is the problem of determining the type of an infinitely-sheeted covering surface of the Riemann sphere with n punctures. Such a covering surface can be represented by a *Speiser graph*, as I will now describe.

Start by drawing a simple closed curve C through the n branch points, as shown in Figure 1. The branch points divide C into n segments, which we label $C_1, \ldots, C_n$. The curve C divides the sphere into two parts, which we label S_a and S_b. Cutting along the curves that cover $C_1, \ldots, C_n$ separates the covering surface into an infinite number of pieces, some that cover S_a and some that cover S_b.

To reconstruct the surface, we must glue each copy of S_a along each of the n curves that form its boundary to one or another of the copies of S_b. The Speiser graph gives a recipe for carrying out these gluings. It is an infinite graph with vertices labelled a and b and edges labelled by integers between 1 and n. Each edge joins a vertex labelled a to one labelled b. Each vertex has n edges coming into it, labelled $1, \ldots, n$. The vertices labelled a correspond to copies of S_a, and those labelled b to copies of S_b. An edge labelled i indicates

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