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SINGULAR LOCUS OF A SCHUBERT VARIETY

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In this note we give a characterization of the singular locus of a Schubert variety in the flag variety G/B, where G is a classical group and B is a Borel subgroup of G, or, more generally, for a Schubert variety in G/Q, where Q is a parabolic subgroup of G of *classical type* (cf. [**4, 5**]) of a semisimple algebraic group G. This turns out to be a rather easy consequence of the standard monomial theory, developed in *Geometry of* G/P, I–V (cf. [**9, 6, 3, 4, 7**]).[1] A corollary of this theory is the determination of the ideal defining X in G/B, and our result follows from the Jacobian criterion for smoothness. When $G = \mathrm{SL}(n)$ this characterization takes a particularly nice form (cf. Theorem 1 and Remark 1). To the best of our knowledge, our result is new even for a Schubert variety in a Grassmannian.[2]

Let G be a semisimple algebraic group, which we assume, for simplicity, to be defined over an algebraically closed field of characteristic zero (the following discussion is valid in any characteristic, in fact even over $\mathbf{Z}$). Let T be a maximal torus, B a Borel subgroup containing T, and W the Weyl group of G. Let L be a line bundle on G/B associated to a dominant character χ of T (or B). Let V be the G-module $H^0(G/B, L)$. Recall that V can be identified with the set of regular functions f on G such that

$$f(gb) = \chi(b)f(g); \qquad b \in B,\ g \in G.$$

Now V is also a $\mathfrak{G}$-module, where $\mathfrak{G} = \operatorname{Lie} G$. We identify $Y \in \mathfrak{G}$ with the corresponding right invariant vector field D_Y on G and if $v \in V$ corresponds to a function f on G as above, we see that $D_Y f = Yf$.

Let U^- be the unipotent part of the Borel subgroup B^-, opposite to B. We can identify U^- as an affine open subset (called the *opposite big cell*) of G/B containing $e_B \in G/B$ (e_B = the point of G/B corresponding to the coset B). Let Δ (resp. Δ^+) be the set of all (resp. positive) roots. Then recall that

$$U^- = \prod_{\alpha \in \Delta^+} U_{-\alpha}, \qquad U_{-\alpha} \simeq \mathbf{G}_a. \tag{1}$$

We denote by $\{x_{-\alpha}\}$, $\alpha \in \Delta^+$, the canonical coordinate system for U^- given by (1). Now we note that if $f \in V$ as above, and we let f be the restriction of f (identified with a function on G as above) to U^-, then the evaluations of $\partial f/\partial X_{-\alpha}$ and $X_{-\alpha} f$ at e_B coincide.

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[2] In a recent talk at Tata Institute, Bombay, Abhyankar gave a class of singular Schubert varieties in $\mathrm{SL}(n)/B$.