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SIMPLE CLOSED GEODESICS ON $H^+/\Gamma(3)$ ARISE FROM THE MARKOV SPECTRUM

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1. Let

$$H^+ = \{z = x + iy \colon y > 0\}$$

be the complex upper half-plane, and let

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n};\ a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}$$

be the principal congruence subgroup of level n in the modular group $\mathrm{SL}(2, \mathbf{Z}) = \Gamma(1)$. In this note we are concerned with $\Gamma(3)$. Let S be the Riemann surface $H^+/\Gamma(3)$ and let $\pi\colon H^+ \to S$ be the projection map. S is a sphere with four punctures.

A hyperbolic element γ is a Möbius transformation of H^+ that has two real fixed points; its axis A_γ is the circle with center on $\mathbf{R}$ connecting the fixed points. Write ξ_γ, ξ'_γ for the fixed points of γ. If $\gamma \in \Gamma(3)$ is hyperbolic, A_γ projects to a closed geodesic on S; conversely, every closed geodesic on S arises in this way. A *simple* closed geodesic is one that does not intersect itself.

The Markov Spectrum will be described in detail in §2. Here we note the definition of the Markov function $M(\theta)$. For real irrational θ set

$$M(\theta) = \sup\{c > 0 \colon |\theta - p/q| < 1/cq^2 \text{ for infinitely many reduced fractions } p/q\}. \tag{1.1}$$

In the range $M(\theta) < 3$, M assumes only a denumerably infinite set of values $M_\nu \uparrow 3$. The numbers M_ν constitute the Markov Spectrum, which we denote by MS.

The connection between simple closed geodesics on S and MS is established in the following way. For $\beta \in \Gamma(3)$ write $A_\gamma \wedge \beta A_\gamma$ to mean $A_\gamma \cap \beta A_\gamma | \neq | \varnothing, A_\gamma$, i.e., the intersection is a single point in H^+. The following criterion is easy to prove:

(1.2) $\pi(A_\gamma)$ is nonsimple if and only if $A_\gamma \wedge \beta A_\gamma$ for some $\beta \in \Gamma(3) - \langle\gamma\rangle$.

But in this statement we know nothing about β except that if is not elliptic ($\Gamma(3)$ contains no elliptic elements).

THEOREM 1. *If $\pi(A_\gamma)$ is nonsimple, there is a parabolic element P in $\Gamma(3)$ such that $A_\gamma \wedge PA_\gamma$.*

Theorem 1 leads directly to the main result:

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