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INTEGRAL REPRESENTAIONS ON HERMITIAN MANIFOLDS: THE $\bar{\partial}$-NEUMANN SOLUTION OF THE CAUCHY-RIEMANN EQUATIONS[1]

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1. Introduction. Let D be a relatively compact domain in a Hermitian manifold X of complex dimension n. The Cauchy-Riemann operator $\bar{\partial}$ extends to a densely defined operator

$$\bar{\partial}\colon L^2_{0,q}(D) \to L^2_{0,q+1}(D), \qquad 0 \le q \le n.$$

The inner product in $L^2_{0,q}(D)$ is given by

$$(f,g)_D = \int_D f \wedge *\bar{g},$$

where $*$ is the Hodge operator defined by the Hermitian structure. If $\bar{\partial}^*$ is the Hilbert space adjoint of $\bar{\partial}$, one defines the complex Laplacian by

$$\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Its significance for complex analysis lies in the fact that if $Nf \in \operatorname{dom}\Box$ solves $\Box(Nf) = f$ and $\bar{\partial} f = 0$, then $u = \bar{\partial}^* Nf$ is the unique solution of $\bar{\partial} u = f$ which is orthogonal to $\ker\bar{\partial}$. J. J. Kohn has established existence and regularity properties of the solution operator N, giving the solution of minimal norm—the so called $\bar{\partial}$-Neumann operator—in case D is strictly pseudoconvex [**5**], and in more general cases as well [**6**]. The proofs are based on a priori estimates in L^2-Sobolev spaces, and therefore they do not give any explicit information about the kernels of N or $\bar{\partial}^*N$.

In recent years there has been much interest in finding more explicit and concrete representations of the abstractly defined operators N and $\bar{\partial}^*N$ (see [**2, 3, 9, 10, 12**]). In [**7**] we began to study $\bar{\partial}^*N$ by using the calculus of Cauchy-Fantappié kernels in $\mathbf{C}^n$, in analogy to the work of Kerzman and Stein [**4**] and Ligocka [**8**] for the Szegö, respectively, Bergman kernel; in contrast to the scalar case, the incompatibility of the Euclidean metric with the complex geometry of the boundary of D turned out to be a major obstruction in the general case.

In the present paper we overcome this obstruction by generalizing the results in [**7**] to arbitrary Hermitian manifolds; this enables us to then introduce a special Levi metric—similar to the one in [**2**]—and to establish the required symmetry properties of the kernels. Our main result gives a new and completely explicit integral representation of the principal part of $\bar{\partial}^*N$ on

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