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WEIGHTED POLYNOMIALS ON FINITE AND INFINITE INTERVALS: A UNIFIED APPROACH

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1. Introduction. As described in the survey article [**6**], the study of "incomplete polynomials", as introduced by G. G. Lorentz [**4**] in 1976, leads to results on the asymptotic properties of polynomials orthogonal on an infinite interval (cf. [**5**]) and to theorems on the convergence of "ray sequences" of Padé approximants for Stieltjes functions. Here we present a generalization of the theory for incomplete polynomials which unifies many of the previous results. The essential question which serves as the starting point for the investigation is the following:[1]

Suppose $w(x)$ is a nonnegative weight function continuous on its support $\Sigma \subset \mathbf{R} = (-\infty, \infty)$. (By the *support* of w we mean the *closure* of the set where w is positive.) Assume that $w(x)$ vanishes at points of Σ; that is, $Z := \{x \in \Sigma : w(x) = 0\} \neq \emptyset$ (or, in case Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$). If P_n is an arbitrary polynomial of degree at most n, then the sup norm over Σ of the weighted polynomial $[w(x)]^n P_n(x)$ actually "lives" on some compact set $S \subset \Sigma - Z$ which is independent of n and P_n. The question is to determine the smallest such set S.

For example, if $w(x) = x^{\theta/(1-\theta)}$ with $\Sigma = [0, 1], 0 < \theta < 1$, then, as shown in [**2, 8**], S is the subinterval $[\theta^2, 1]$.

In this paper we use potential theoretic methods to show how S can be obtained for a class of weight functions. The assumptions on w are given in

DEFINITION 1.1. Let $w\colon \mathbf{R} \to [0, +\infty)$. We say that w is an *admissible weight function* if each of the following properties holds:

(i) $\Sigma := \operatorname{supp}(w)$ has positive capacity.

(ii) The restriction of w to Σ is continuous on Σ.

(iii) The set $Z := \{x \in \Sigma\colon w(x) = 0\}$ has capacity zero.

(iv) If Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.

Here, and throughout the paper, the term "capacity" means inner logarithmic capacity (cf. [**10**, p. 55]). For any set $E \subset \mathbf{R}^2$, its capacity will be denoted by $C(E)$. If K is a compact set with positive capacity, then ν_K denotes the unique unit equilibrium measure on K with the property that (cf. [**10**, p. 60])

$$\int_K \log|x - t|\, d\nu_K(t) = \log C(K) \tag{1.1}$$

quasi-everywhere (q.e.) on K. (A property is said to hold q.e. on a set A if the subset E of A where it does not hold satisfies $C(E) = 0$.)

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