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SECONDARY CLASSES AND TRANSVERSE MEASURE THEORY OF A FOLIATION

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1. The purpose of this note is to announce several theorems showing how the secondary classes of a foliation $\mathcal{F}$ of a compact manifold X depend upon the measure theoretic properties of the equivalence relation determined by the foliation. The relevant properties are:

(i) amenability [**14**], which is equivalent to hyperfiniteness by Connes-Feldman-Weiss [**3**]; and

(ii) the Murray-von Neumann type.

A set $B \subset X$ is *saturated* if it is the union of leaves of $\mathcal{F}$. The equivalence relation $\mathcal{F}$ has *type* I if there is a measurable subset of X which intersects almost every leaf exactly once; *type* II if it admits an invariant measure, finite or infinite in the given measure class but does not have an essential saturated set of type I; and *type* III if it does not have any essential saturated sets of types I or II. Every equivalence relation can be decomposed into parts of types I, II, and III. These types correspond to certain algebraic properties of the von Neumann algebra $\mathcal{M}(X, \mathcal{F})$ associated with the equivalence relation [**1, 13**].

Let X be a compact manifold without boundary and $\mathcal{F}$ a C^2, codimension-n foliation of X. The secondary classes are given by a map $\Delta_*\colon H^*(\mathrm{WO}_n) \to H^*(X; \mathbf{R})$ with image spanned by the classes of the form $\Delta_*(y_I c_J)$. Here, y_I is a basis element for the relative cohomology $H^*(\mathrm{gl}_n, \mathrm{O}_n)$, and c_J is a Chern form of degree at most $2n$. If degree $c_J = 2n$, we say the class is *residual.* The Godbillon-Vey classes are those of the form $\Delta_*(y_1 c_J) \in H^{2n+1}(X; \mathbf{R})$, with $y_1 \in H^1(\mathrm{gl}_n, \mathrm{O}_n)$, the normalized basis element. The generalized Godbillon-Vey classes are those of the form $\Delta_*(y_1 y_I c_J)$, where $y_I = 1$ is permitted. (For a convenient reference, see [**11**].)

The residual secondary classes have the unusual property that they localize to the measurable saturated subsets of X: for each such $B \subset X$ and residual $y_I c_J \in H^p(\mathrm{WO}_n)$, the restriction $\Delta_*(y_I c_J)|B \in H^p(X)$ is well defined [**5**]. The following theorems are stated for the secondary classes of $\mathcal{F}$ on X, but corresponding theorems also hold for the localized classes $\Delta_*(y_I c_J)|B$ of the restriction $\mathcal{F}|B$.

THEOREM 1. *If $\mathcal{F}$ has type* I, *then all residual secondary classes of $\mathcal{F}$ are zero.*

Since $\mathcal{F}$ has type I if and only if it is a fibration in the category of measurable equivalence relations, Theorem 1 generalizes the well-known fact that the secondary classes are zero for a smooth fibration.

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