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THE METRIC ENTROPY OF DIFFEOMORPHISMS

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0. Introduction. Let M be a C^∞ compact Riemannian manifold without boundary, let f be a C^2 diffeomorphism of M, and let m be an f-invariant Borel probability measure on M. The entropy of f, $h_m(f)$, measures the amount of randomness with respect to m generated by iterating f. In this note we announce some results concerning the relationship between $h_m(f)$ and the geometric properties of f and m. Our main results are the characterization of measures satisfying Pesin's entropy formula (Theorem 1) and the generalization of this formula to one that applies to all invariant measures (Theorem 2). This generalized formula involves the notion of dimension and leads to other volume estimates (Theorem 3, Corollaries 4 and 5).

1. Preliminaries. By a theorem of Oseledecs [**O**], for m-a.e. x there are numbers $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ and a decomposition of the tangent space T_xM at x into $T_xM = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for every nonzero vector $v \in E_i(x)$, $(1/n)\log\|Df_x^n v\| \to \lambda_i(x)$ as $n \to \pm\infty$.

For x and i with $\lambda_i(x) > 0$, let

$$W^i(x) = \left\{ y \in M\colon \liminf_{n\to\infty} -\frac{1}{n}\log d(f^{-n}x, f^{-n}y) \geq \lambda_i(x) \right\}.$$

Then at a.e. x, $W^i(x)$ is an immersed copy of Euclidean space, the tangent space to $W^i(x)$ at x being $\bigoplus_{j\leq i} E_j(x)$ [**Ru**]. If $u(x) = \max\{i\colon \lambda_i(x) > 0\}$, then $W^{u(x)}(x)$ is called the unstable manifold at x.

Recall that if ξ is a measurable partition of M, then there is a family of conditional measures $\{m_x^\xi\}$ defined at m-a.e. x such that the support of m_x^ξ lies in $\xi(x)$, the element of ξ containing x, and $m(A) = \int m_x^\xi(A)\, m(dx)$ for every measurable set $A \subset M$. (See [**Ro**].)

Whenever $W^i(x)$ is defined and is an immersed submanifold, the Riemannian structure on M induces a Riemannian structure and, hence, a Riemannian measure on $W^i(x)$. We denote this measure by μ_x^i. A measurable partition ξ is said to be *subordinate* to $W^{u(\cdot)}$ if for m-a.e. x, $\xi(x) \subset W^{u(x)}(x)$ and contains an open neighborhood of x in $W^{u(x)}(x)$. Partitions adapted to W^i are defined similarly. We say that m has *absolutely continuous conditional measures on unstable manifolds* if for every ξ subordinate to $W^{u(\cdot)}$, m_x^ξ is absolutely continuous to $\mu_x^{u(x)}$ at a.e. x.

We omit in the statements of our theorems the standing hypotheses that $f\colon M \to M$ is a C^2 diffeomorphism and m is an f-invariant Borel probability measure on M.

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