BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 2, October 1984

COMPLEXIFICATION, TWISTOR THEORY, AND HARMONIC MAPS FROM RIEMANN SURFACES

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ABSTRACT. Penrose's *twistor theory* and many other ideas of mathematical physics are based on the notion of *complexification*. This notion is explained and examples of its application in physics and mathematics are described. In particular, the well-known analogy between Yang-Mills fields and harmonic maps of Riemann surfaces becomes rather stronger after complexification. This strengthening is the main point of this paper.

Introduction. Throughout Penrose's development of twistor theory [**25, 28**], complexification is omnipresent, albeit often only implicitly. This technique is probably more familiar to physicists than mathematicians, especially its informal use in quantum field theory. It is, however, a quite precise construction which can probably find greater application in mathematics than it presently enjoys. The idea is that many structures based on real numbers become more understandable if viewed in a complex environment. Familiar examples are algebraic varieties and trigonometric or elliptic functions. In §1 a brief exposition of the complexification of real-analytic structures is given, together with a few examples.

The main examples of this article, however, come from twistor theory and the theory of harmonic maps as discussed in §§2 and 3 respectively. As regards twistor theory, the Ward correspondence for self-dual Yang-Mills fields emerges as a direct analogue of the Cauchy-Riemann equations for Riemann surfaces. The harmonic maps of §3 will be from a Riemann surface into complex projective space. The construction of such mappings due to Din and Zakrzewski [**7, 8**] and, independently, Burns [**4**] (see also Eells and Wood [**11, 12**]) becomes rather clearer after complexification. This clarification and the strengthening of the well-known analogy between Yang-Mills fields and harmonic maps are two of the aims of this article. Also, this gives an illustration of the utility of complexification.

This utility is well known to those who know and in particular to Roger Penrose and Claude LeBrun (see [**18–20**]), to whom I am grateful for many useful conversations. I would also like to thank the Institut des Hautes Études Scientifiques for hospitality during the work described in this paper.

1. Complexification. Suppose M is a real-analytic manifold. Each coordinate patch $U \subset \mathbb{R}^n$ can be enlarged to a neighbourhood $\mathbb{C}U \subset \mathbb{C}^n$. Since the coordinate changes are real analytic functions, they may be extended to these

Received by the editors January 4, 1984.
1980 *Mathematics Subject Classification*. Primary 53B35; Secondary 53B50.
[1]Research supported by an SERC Advanced Fellowship.

0273-0979/84 $1.00 + $.25 per page