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Semimartingales, a course on stochastic processes, by Michel Métivier, de Gruyter Studies in Mathematics, Vol. 2, Walter de Gruyter & Co., Berlin, 1982, xi + 287 pp., DM 88.--, $40.00. ISBN 3-1100--8674-3

The semimartingale calculus has emerged from the general theory of processes as an important tool for what Métivier claims in his preface is "the goal of many: the description of stochastic systems, the ability to study their behavior and the possibility of writing formulas and computational algorithms to evaluate and identify them (without mentioning their optimization!)." We will first describe some important ingredients of the calculus, beginning with the martingale stochastic integral.

Let $(\Omega, \mathbf{F}, P)$ be a probability space and let $(\mathbf{F}_t\colon t \geqslant 0)$ be an increasing family of sub-σ-algebras of $\mathbf{F}$. A random process $M = (M_t)$ is a martingale if M_s is an $\mathbf{F}_s$ measurable random variable for each s (i.e., if M is adapted) and if $E[M_t|\mathbf{F}_s] = M_s$ whenever $t \geqslant s$. This definition makes sense even for Banach-space-valued random processes. On the other hand, it is rather tricky to define martingales with values in a manifold—but J. M. Bismut did it using localization and manifold connections [**22**]. The most famous martingale is the Wiener process, sometimes called brownian motion, $W = (W_t\colon t \geqslant 0)$. (See [**24**] for references omitted here.)

A stochastic integral

$$\int_0^t h(s, \omega)\, dM(s, \omega) \qquad \left(\text{abbreviated } \int_0^t h_s\, dM_s\right)$$

can be defined when M is a martingale for certain types of functions h. As the notation suggests, a stochastic integral is similar to a Stieltjes-Lebesgue integral for each ω in Ω. However, the notation is misleading—many interesting martingales, the Wiener process included, have sample paths of unbounded variation over any interval for ω in a set of probability one. This precludes defining the stochastic integral as a Stieltjes-Lebesgue integral for each ω.

A recipe for contructing stochastic integrals is the following. First, if h is piecewise constant in t with jumps at $t_1, t_2, \ldots, t_n$, it is natural to define the stochastic integral by

$$\int_0^t h_s\, dM_s = \sum_i h(t_i)\left(M_{t_{i+1}} - M_{t_i}\right).$$

If the value of the integral, which is a random variable for each t, is sufficiently continuous in h as h ranges over a suitable collection of step functions, then the integral can be defined for h in a larger collection of functions.

Stochastic integrals were first constructed by N. Wiener. He considered the case in which M is a Wiener process and h is a function of s alone. The above recipe works for constructing the Wiener integral, although Wiener used a