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DEFINABLE SETS IN ORDERED STRUCTURES

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1. Introduction. We introduce the notion of an O-minimal theory of ordered structures, such a theory being one such that the definable *subsets* of its models are particularly simple. The theory of real closed fields will be an example. For T an O-minimal theory we prove that over every subset A of a model there is a prime model, which is unique up to A-isomorphism. We also prove in our model-theoretic context results on the structure of semialgebraic sets. Our work was directly stimulated by the paper of van den Dries [**4**].

2. Definitions and examples. L will be a finitary first order language which contains, among other things, a symbol $<$. We shall be concerned with infinite L-structures M in which $<$ denotes a linear ordering of M. For example if L has symbols $<$, $+$, 0, then an ordered group is just an L-structure G which satisfies the axioms for ordered groups. A *definable subset* of M^n is a subset $X \subset M^n$ of the form $\{\overline{a} \in M^n : M \models \varphi(\overline{a}, \overline{m})\}$ for some L-formula $\varphi(\overline{x}, \overline{y})$ and $m \in M^r$, $r < \omega$. So the definable sets are those which are obtained from the sets defined with parameters from the basic relations and functions on M, by closing under finite unions, finite intersections, complementation and projection. An interval of M is something of the form $(a,b), [a,b], (a,b]$ or $[a,b)$, where $a, b \in M$ (or $a = -\infty$, or $b = +\infty$). (Such an interval is clearly definable.)

DEFINITION 1. (i) M is *O-minimal* if every definable set $X \subset M$ is a finite union of rational intervals of M.

(ii) A complete L-theory T is O-minimal if every model of T is O-minimal.

Note that in Definition 1 no condition is placed on the definable subsets of M^n. An important consequence of the definition is that an O-minimal structure is *definably complete*; namely every definable subset of M which is bounded above has a supremum in M (and similarly for infimums).

A consequence of the Tarski-Seidenberg theory [**3**] (i.e. quantifier elimination), is that any real closed field is O-minimal. In fact if K is a real closed field then the definable subsets of K^n, $n < \omega$, are precisely the semialgebraic sets over K. The following is proved, essentially using the "definable completeness" of O-minimal structures:

THEOREM 2. (i) *An ordered group G is O-minimal just if G is abelian and divisible.*

(ii) *An ordered unitary ring R is O-minimal just if R is a real closed field.*

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